Towards a Gluing Construction for Constant Mean Curvature Surfaces Along Their Ends

Jesse Ratzkin

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1 Introduction and Notation

The general situation we will consider is the following. We have an immersion $X : \Sigma \to \mathbb{R}^3$ of a surface with finite topology. We will assume that the induced metric on Σ is complete. If (s, θ) are coordinates on Σ, we will denote the induced metric by $g = Eds^2 + 2Fdsd\theta + Gd\theta^2$, the Gauss map as $\nu = \frac{\partial_s X \times \partial_\theta X}{|\partial_s X \times \partial_\theta X|}$, and the second fundamental form as $A = Lds^2 + 2Mdsd\theta + Nd\theta^2$. We denote the Gauss curvature by $K = \det A$ and the mean curvature as $H = \frac{1}{2} \operatorname{tr}_g A$. Notice that A is an extrinsically defined object; it depends on the immersion X. It turns out that K is a Riemannian invariant of Σ. This is essentially Gauss' *Theorem Egregium.* On the other hand, the mean curvature H is not a Riemannian invariant. For instance, a flat infinite strip of width π in the plane plane and half of a cylinder of radius $\frac{1}{2}$ are isometric, but the strip has zero mean curvature while the cylinder has mean curvature 1.

Here we will study those immersions which have mean curvature identically 1. These immersions are called CMC immersions for short. Notice that if the mean curvature is any other nonzero constant, then a homothety and possible reversal of orientation will yield an immersion with mean curvature 1. So immersions with constant mean curvature fall into two categories: those with mean curvature zero (minimal immersions) and those with mean curvature one (CMC immersions). Recent years have seen many advances in the classification of both CMC and minimal immersions. The surfaces we will concentrate on here are complete, noncompact, CMC embeddings of finite topology. Results of the last 15 years include theorems describing the asymptotic structure of the ends of such surfaces (see [17]) and the regularity of the moduli spaces of such surfaces (see [20]).

The results mentioned above are in some sense the sort of results common in modern, twentieth-century geometric analysis. They are general uniqueness and classification results. Another small industry in this area is the production of examples, which is in some sense a typical nineteenth-century pursuit. The unit sphere $\mathbb{S}^2$ and the cylinder of radius $\frac{1}{2}$ are basic examples of CMC surfaces. In 1841 Delaunay (see [5]) found a one-parameter family of CMC surfaces which are all rotationally symmetric and interpolate between the cylinder and a string of spheres which lie tangent to each other (see section 3.1 for a description of these surfaces). In contrast to the theory of minimal surfaces, it took a long time to construct more examples of CMC surfaces.

Indeed, the next examples of embedded CMC surfaces did not appear until Kapouleas constructed them in 1990 in [16]. (The history of compact CMC surfaces which are immersed but not embedded is equally rich, but very different.) See section 3.3 for a description of his construction. In [18] and [19] Mazzeo, Pacard, and Pollack solved nonlinear boundary value problems to glue either Delaunay ends onto a k-noid with catenoid ends ([18]) or to glue two CMC surfaces together with a catenoid neck. In these results, they found an explicit Greens kernel for the linear operator and then matched the Cauchy data. The gluing method used here is to build the approximate solution by patching together known solutions with cut-off functions. Instead of the singular perturbation theory of [16] we will use a better understanding of solutions to the linearized mean curvature operator. This understanding is similar to the analysis Melrose uses in [10] to study harmonic forms on compact manifolds with boundary. The guiding principle is that tempered solutions (solutions with subexponential growth) to the linearized operator lie in a finite dimensional vector space which arises from geometric deformations of the ends (e.g. translations and rotations of the ends). We then use solutions to the linearized mean curvature operator in this finite dimensional space to adjust the approximate solution.

The current project is to glue two CMC surfaces together "end-to-end" in a sense to be described below. We start with two noncompact complete CMC embedded surfaces Σ_1 and Σ_2 of finite topology. The ends of Σ_i are the unbounded connected components of $\Sigma_i \backslash \mathbb{B}_r(0)$, where r is taken large enough so that the number of such components is constant. Pick ends E_i of Σ_i. By a result of Korevaar, Kusner, and Solomon (see section 4.3) one can understand the asymptotic structure of the ends E_i. We require that the asymptotic structure of E_1 matches that of E_2. Align Σ_1 and Σ_2 such that E_1 and E_2 lie along the same axis, but point in opposite directions. One can then patch Σ_1 and Σ_2 together using a cut-off function to get a surface $\bar{\Sigma}$ which has mean curvature 1 away from the patching region and has mean curvature close to 1 in the patching region. The goal is to now perturb $\bar{\Sigma}$ and find a nearby CMC surface Σ. We will describe this in more detail in Section 6. The purpose of this project is to construct new families of CMC surfaces.

Finally, we should remark that the theory of complete noncompact CMC immersions has many similarities to the theory of metrics constant positive scalar curvature. Indeed, as noted in [22] and [20], each theorem regarding one problem seems to have a counterpart with the other problem. For instance, the results about the moduli spaces of k-ended CMC surfaces and the moduli spaces of singular Yamabe metrics on $\mathbb{S}^n$ (complete metrics of constant positive scalar curvature on $\mathbb{S}^n \backslash \{p_1 \dots p_k\}$ conformal to the usual metric) have remarkably similar statements (compare the statements of Theorem 1.4 of [22] and Theorem 1.3 of [20]). In fact, most of the analysis for the CMC problem carries over to the CPSC problem.

2 Various Formulations of the CMC Condition

One can formulate the condition that an immersion is of constant mean curvature in various ways. Each is useful to understand some part of the general theory of CMC surfaces.

2.1 The Local Formulation: Principal Curvatures

We start with the local formulation in terms of coordinates (s, θ) on Σ. Then

$$g = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \langle X_s, X_s \rangle & \langle X_s, X_\theta \rangle \\ \langle X_s, X_\theta \rangle & \langle X_\theta, X_\theta \rangle \end{bmatrix},$$

$$A = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \langle X_{ss}, \nu \rangle & \langle X_{s\theta}, \nu \rangle \\ \langle X_{s\theta}, \nu \rangle & \langle X_{\theta\theta}, \nu \rangle \end{bmatrix},$$

and

$$H = \frac{1}{2}\operatorname{tr}_g A = \frac{1}{2}\frac{LG + NE - 2FM}{EG - F^2}.$$

Near any point, we can write Σ as a graph over its tangent plane. Then the immersion X takes the form

$$X(s,\theta) = (s, \theta, f(s,\theta)).$$

If X takes this form, the metric is given by

$$g = \begin{bmatrix} 1 + f_s^2 & f_s f_\theta \\ f_s f_\theta & 1 + f_\theta^2 \end{bmatrix}$$

and the second fundamental form is given by

$$A = \frac{1}{\sqrt{1 + f_s^2 + f_\theta^2}} \begin{bmatrix} f_{ss} & f_{s\theta} \\ f_{s\theta} & f_{\theta\theta} \end{bmatrix}.$$

In particular, the mean curvature is given by

$$H = \frac{f_{ss}(1 + f_\theta^2) + f_{\theta\theta}(1 + f_s^2) - 2 f_{s\theta} f_s f_\theta}{2(1 + f_s^2 + f_\theta^2)^{\frac{3}{2}}}.$$

Setting $H = 1$ and rearranging yields

$$0 = f_{ss}(1 + f_\theta^2) + f_{\theta\theta}(1 + f_s^2) - 2 f_{s\theta} f_s f_\theta - 2(1 + f_s^2 + f_\theta^2)^{\frac{3}{2}}. \tag{1}$$

Several remarks on equation (1) will prove useful. First, this is a quasilinear second order PDE in f. It is strongly elliptic. In fact, the linearization of the second order part the right hand side of equation (1) is given by the matrix

$$\begin{bmatrix} 1 + f_\theta^2 & -f_s f_\theta \\ -f_s f_\theta & 1 + f_s^2 \end{bmatrix}.$$

And so the principal symbol of equation (1) is given by

$$\begin{bmatrix} \lambda & \mu \end{bmatrix} \begin{bmatrix} 1 + f_\theta^2 & -f_s f_\theta \\ -f_s f_\theta & 1 + f_s^2 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \lambda^2 + \mu^2 + (\lambda f_\theta - \mu f_s)^2 \geq 0$$

with equality only when $\lambda = 0 = \mu$. This implies, among other things, that CMC surfaces are analytic (elliptic regularity) and the function f obeys the strong maximum principle (e.g . f can have no positive interior maxima; see, for example [12] or [13]).

One can also attach a geometric interpretation to these coordinate computations. In the following paragraph, we will work only at the origin in the (s,θ) coordinates and we will assume that

$$0 = f(0,0) = f_s(0,0) = f_\theta(0,0).$$

This amounts to setting the (s, θ) plane to be the tangent plane to Σ at the point corresponding to $(0,0)$. Then

$$g(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A(0,0) = \begin{bmatrix} f_{ss} & f_{s\theta} \\ f_{s\theta} & f_{\theta\theta} \end{bmatrix}.$$

Recalling the minimax method to find eigenvalues using the Raleigh quotient, we see that the eigenvectors of A point in the direction of steepest descent and ascent for the function f. Call these eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Order them so that their respective eigenvalues k_1 and k_2 satisfy $k_1 \leq k_2$. Notice that setting $k_1 + k_2 = 2$ and $k_1 \leq k_2$ implies $k_2 \geq 1 > 0$. If we let $h(t) = f(t\vec{v}_2)$ the we see $t = 0$ is a local minimum for h. In fact, the graph of h is concave up and the circle lying above the graph which best fits the graph will have radius $\frac{1}{h''(0)} = \frac{1}{k_2}$. Similar remarks hold for k_1, although one must be careful of signs when treating this case. Thus we see that the eigenvalues k_1 and k_2 correspond to radii of the largest and smallest circles (taking signs into account) fitting curves in Σ one finds by intersecting Σ with a plane normal to Σ at the origin. These eigenvalues k_1 and k_2 are called the principal curvatures of Σ and the eigendirections span $\vec{v}_1$ and span $\vec{v}_2$ are called the principal direction. A point on Σ is called umbilic if $k_1 = k_2$. The preceding discussion shows that the mean curvature of a surface at a point p is the average of the curvature of curves in Σ through p in all directions, giving credence to the name "mean curvature".

2.2 The Variational Formulation

The variational set-up described below is the same as in [14]. This formulation of the CMC condition is classical. One can find a modern treatment of it in volume IV of [15] (towards the end of Chapter 9) and [8].

On can also formulate the condition that X is a CMC immersion in variational terms. First consider a one parameter family of immersions $X_t : \Sigma \to \mathbb{R}^3$ with $X_0 = X$. Then the first variation of area $\frac{d}{dt}\big|_{t=0} \operatorname{Area}(X_t(\Sigma)) = \frac{d}{dt}\big|_{t=0} \int_\Sigma X_t^*(dV)$ is given by

$$\frac{d}{dt}\bigg|_{t=0} \operatorname{Area}(X_t(\Sigma)) = \int_\Sigma \langle \frac{d}{dt}\bigg|_{t=0} X_t, H\nu \rangle.$$

Now consider the following situation. Let X be a CMC immersion of Σ as above and let $U \subset \mathbb{R}^3$ be a bounded open set with $\partial U = Q \cup S$ where S is an open subset of $X(\Sigma)$ and $\partial Q = \partial S = \Gamma$ is a smooth closed curve in $X(\Sigma)$. Let V be a vector field supported in $U \backslash \bar{Q}$ and denote its flow by ϕ_t. This vector field yields a one parameter family of surfaces $S_t = \phi_t(S)$ and a one parameter family of solids $U_t = \phi_t(U)$. Pick a real constant H and let h denote the mean curvature of X.

Then the formula for the first variation of volume yields

$$\left.\frac{d}{dt}\right|_{t=0}(\mathrm{Area}(S_t) - H\,\mathrm{Vol}(U_t)) = (h-H)\,\mathrm{Area}(S).$$

Thus we see that surfaces with mean curvature identically H are critical points of the functional Area $-H$ Vol.

2.3 The Hopf Differential, the Sinh-Gordon Equation, and Harmonicity of the Gauss Map

Much of this formulation can be found in [9].

For this section we will work in conformal coordinates on Σ. In other words, we will let (s,θ) be coordinates on Σ such that $E = G = 2e^{2\omega}$ and $F = 0$. Then $z = s + i\theta$ is a complex coordinate on Σ. Define the vector fields

$$\partial_z = \frac{1}{2}(\partial_s - i\partial_\theta) \text{ and } \partial_{\bar{z}} = \frac{1}{2}(\partial_s + i\partial_\theta).$$

Notice that

$$\partial_z\partial_{\bar{z}} = \frac{1}{4}\Delta.$$

Consider the immersion X restricted to a simply connected region Ω on the surface. The condition that $z = s + i\theta$ is a conformal coordinate with conformal factor $2e^{2\omega}$ is equivalent to

$$\langle X_z, X_z\rangle = 0 \qquad \langle X_z, X_{\bar{z}}\rangle = e^{2\omega}.$$

In addition, we also have

$$\langle \nu, X_z\rangle = 0 \qquad \langle \nu, X_{\bar{z}}\rangle = 0.$$

Taking derivatives of these equations yields

$$\langle X_{zz}, X_z\rangle = 0 \quad \langle X_{z\bar{z}}, X_z\rangle = 0 \quad \langle X_{zz}, X_{\bar{z}}\rangle = 2\omega_z e^{2\omega}$$

and

$$\langle \nu, X_{zz}\rangle + \langle \nu_z X_z\rangle = 0 \quad \langle \nu, X_{z\bar{z}}\rangle + \langle \nu_{\bar{z}}, X_z\rangle = 0.$$

If we let $\langle \nu, X_{zz}\rangle = Q$ and note $\langle \nu, X_{z\bar{z}}\rangle = \frac{1}{4}\langle \nu, \Delta X\rangle = \frac{1}{2}e^{2\omega}H$, then the above equations imply

$$X_{zz} = 2\omega_z X_z + Q\nu \quad X_{z\bar{z}} = \frac{1}{2}e^{2\omega}H\nu \quad \nu_z = -\frac{1}{2}HX_z - Qe^{-2\omega}X_{\bar{z}}. \tag{2}$$

As a side note, Q is the coefficient of a quadratic differential form Qdz^2. The function Q itself is only locally defined, but Qdz^2 is a globally defined quadratic differential form. This quadratic form is called the Hopf differential.

We can rewrite equations (2) as

$$\begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix}_z = \begin{bmatrix} 2\omega_z & 0 & Q \\ 0 & 0 & \frac{1}{2}e^{2\omega}H \\ -\frac{1}{2}H & -Qe^{-2\omega} & 0 \end{bmatrix} \begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix} = U \begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix}. \tag{3}$$

Similarly,

$$\begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix}_{\bar{z}} = \begin{bmatrix} 0 & 0 & \frac{1}{2}e^{2\omega}H \\ 0 & 2\omega_{\bar{z}} & \bar{Q} \\ -\bar{Q}e^{-2\omega} & -\frac{1}{2}H & 0 \end{bmatrix} \begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix} = V \begin{bmatrix} X_z \\ X_{\bar{z}} \\ \nu \end{bmatrix}. \tag{4}$$

Setting $\partial_{\bar{z}}$ of equation (3) equal to ∂_z of equation (4) yields

$$U_{\bar{z}} - V_z + [U, V] = 0. \tag{5}$$

One can compute that

$$U_{\bar{z}} - V_z + [U,V] = \begin{bmatrix} 2\omega_{z\bar{z}} - |Q|^2e^{-2\omega} + \frac{1}{4}H^2e^{2\omega} & 0 & Q_{\bar{z}} - \frac{1}{2}e^{2\omega}H_z \\ 0 & -2\omega_{z\bar{z}} + |Q|^2e^{-2\omega} - \frac{1}{4}H^2e^{2\omega} & -\bar{Q}_z + \frac{1}{2}e^{2\omega}H_{\bar{z}} \\ -\frac{1}{2}H_{\bar{z}} + e^{-2\omega}\bar{Q}_z & \frac{1}{2}H_z - e^{-2\omega}Q_{\bar{z}} & 0 \end{bmatrix}.$$

Setting this quantity to zero yields the following two equations:

$$\Delta\omega + \frac{1}{2}H^2e^{2\omega} - 2|Q|^2e^{-2\omega} = 0 \tag{6}$$

and

$$Q_{\bar{z}} - \frac{1}{2}e^{2\omega}H_z = 0. \tag{7}$$

Recalling that H is real-valued (and so $H_{\bar{z}} = (\bar{H_z})$), we see that the latter equation implies H is constant if and only if Q is holomorphic. From this Hopf (see [11]) proved

Theorem 1 (Hopf's Theorem)*: Let Σ be a compact simply connected immersed CMC surface. Then Σ is a round sphere.*

First note that we can rewrite Q as

$$Q = \frac{L - N}{2} - iM.$$

From this formulation we conclude that zeroes of the Hopf differential are umbilic points. By uniformization, if Σ is a compact simply connected surface then Σ is conformally equivalent to a sphere. From the fact that Σ is CMC we conclude that Qdz^2 is a holomorphic differential on the sphere. This forces $Qdz^2 = 0$ on all of Σ, and so all points of Σ are umbilic. From this fact it is easy to show that Σ must be a round sphere.

If X is a mean curvature one immersion of a torus, then one can extend the function Q from a small patch Ω about the origin to be a doubly periodic function on the entire plane $\mathbb{C}$. In particular, Q is a bounded holomorphic function on $\mathbb{C}$ and hence must be constant. After multiplying by an appropriate number in the domain, we can choose $Q = \frac{1}{2}$. With $Q = \frac{1}{2}$ and $H = 1$, equation (6) now becomes

$$\Delta\omega + \sinh 2\omega = 0, \tag{8}$$

which is known as the Sinh-Gordon equation. Notice that the rescaling to set $Q = \frac{1}{2}$ is a rescaling in the parameter space and the rescaling to set $H = 1$ is a rescaling in the target space. In particular, these rescalings can be done independently.

Further computation shows

$$\nu_{z\bar{z}} = -[\frac{1}{2}H_{\bar{z}}X_z + Q_{\bar{z}}e^{-2\omega}X_{\bar{z}} + (\frac{1}{4}H^2e^{2\omega} + |Q|^2e^{-2\omega})\nu].$$

Thus X is a CMC immersion if and only if $\nu_{z\bar{z}}$ is a multiple of ν. Recalling that $\nu : \Omega \to \mathbb{S}^2$, we see that $\Delta\nu = \lambda\nu$ is precisely the condition that ν is a harmonic map into $\mathbb{S}^2$. Thus X is a CMC immersion if and only if the Gauss map ν is harmonic.

3 Examples

As mentioned above, the unit sphere and the cylinder of radius $\frac{1}{2}$ are both CMC. The Delaunay surfaces provide the next example of embedded CMC surfaces. One can think of these surfaces as interpolating between spheres and cylinders.

3.1 Delaunay Surfaces

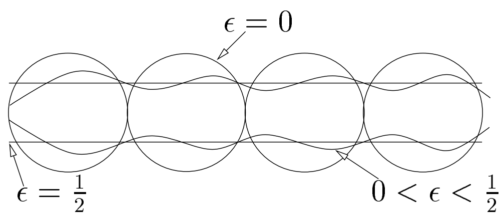
We seek an embedding of the form

$$D(t, \theta) = (\rho(t)\cos\theta, \rho(t)\sin\theta, t) : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^3$$

with mean curvature 1. An embedding of this form is rotationally symmetric about the z axis. The condition that D is an embedding implies $\rho > 0$. The CMC condition implies that ρ satisfies the equation

$$\rho_{tt} - \frac{1}{\rho}(1 + \rho_t^2) + 2(1 + \rho_t^2)^{\frac{3}{2}} = 0. \tag{9}$$

One particular solution is $\rho = \frac{1}{2}$. This solution corresponds to the cylinder. Normalize ρ so that ρ assumes a local minimum of ϵ at $t = 0$ (this amounts to a translation in the t variable). One can then show that ρ is periodic and in fact ϵ is a global minimum for ρ. Critical points for ρ alternate between minima and maxima. The minimum value (ϵ) for ρ is called the necksize of the embedding. One can show that as $\epsilon \to 0$ the embedding D tends to a string of unit spheres $\{x^2 + y^2 + (z - 2n)^2 = 1\}$ for $n \in \mathbb{Z}$.



We will change variables, first to make D into a conformal embedding. To this end, we must replace t with $k(s)$ where k satisfies the equation

$$\rho(k(s)) = k'(s)(\rho'(k(s)) + 1).$$

Now let $\tau = 2\epsilon - \epsilon^2$ and define $\sigma(s)$ by $\rho(k(s)) = \tau e^{\sigma(s)}$. One can show that τ is a first integral of equation (9), see section 4.2. Then one can show

$$\sigma_{ss} + \frac{\tau^2}{2}\sinh 2\sigma = 0 \qquad \frac{dk}{ds} = \frac{\tau^2}{2}(e^{2\sigma} + 1).$$

In fact, finding solutions to the above equations is equivalent to finding an embedded Delaunay surface.

Geometrically, one can think of the Delaunay surfaces as interpolating between the cylinder and the string of spheres. First place an ellipse tangent to the z axis in the $x - z$ plane so that one of the foci is on the x axis and is positioned to that it is as close to the z axis as possible. Now roll the ellipse along the z axis. One can show that the focal point which started on the x axis traces out the profile curve of a Delaunay surface (see [23]). Varying the eccentricity of the ellipse corresponds to varying the necksize of the surface. The cylinder corresponds to rolling a circle of radius $\frac{1}{2}$ (the center is the only focal point and stays at constant height $\frac{1}{2}$). The string of spheres corresponds to rolling a line segment of length 1 (this is the degenerate case where the eccentricity goes to ∞ and the focal points go to the endpoints of the line segment).

3.2 CMC Tori

One might think to look for CMC immersions of compact surfaces. In the 1950's Hopf proved that any simply connected CMC immersion of a compact surface has to be a round sphere (see Hopf's theorem above, or [11]). Around the same time, Alexandrov proved that any embedded compact CMC surface must be the round sphere (see Theorem 3 below). If one were looking for compact CMC immersions, then given these two results one might next look for CMC tori. Below we will regard a torus as $\mathbb{R}^2/\Gamma$ where Γ is a lattice.

To find CMC tori, we look for doubly periodic immersions $\mathbb{R}^2 \to \mathbb{R}^3$. We can reduce this problem as follows. First, note that any immersion of a surface is determined up to rigid motions by its metric and its second fundamental form. Also notice that locally, the metric is determined by its conformal factor and that equations (3) and (4) determine the second fundamental form. Therefore, the conformal exponent ω will locally determine the immersion. Finally, notice that if Σ is a torus then ω must in fact be a doubly periodic function on $\mathbb{R}^2$. Thus the task of finding a CMC torus is the same as finding a doubly periodic solution to the Sinh-Gordon equation (equation (8)). In 1986, Wente proved [2] that such doubly periodic solutions exist.

In 1987, Aubresch ([3]) found many CMC tori by requiring that one line of curvature be planar. The condition that A has distinct eigenvalues allows us to simultaneously diagonalize A and g, so away from umbilic points we can choose coordinate lines which are also lines of curvature. The condition that the θ coordinate line in planar is equivalent to

$$\omega_{s\theta} \cosh \omega - \omega_s \omega_\theta \sinh \omega = 0.$$

We combine this equation with equation (8) to get an overdetermined system of equations. Under the change of variables $W = \cosh \omega$ this system becomes

$$\left\{ \begin{array}{rcl} (W^2-1)\Delta W - W|\nabla W|^2 + W(W^2-1)^2 & = & 0 \\ (W^2-1)W_{s\theta} - 2WW_sW_\theta & = & 0. \end{array} \right.$$

Theorem 2 (Aubresch): *The real analytic solutions of the above system are given by*

$$W = \frac{f_s + g_\theta}{1 + f^2 + g^2}$$

where $f(s)$ and $g(\theta)$ are elliptic functions. Moreover, one can recover f and g by

$$\left\{ \begin{array}{rcl} W_s & = & -f(s)(W^2-1) \\ W_\theta & = & -g(\theta)(W^2-1). \end{array} \right.$$

However, one still has to find conditions so that W is doubly periodic (these are called closing conditions). This is a rationality condition on the initial conditions c and d of f and g. Thus the CMC tori with one planar line of curvature are parameterized by the two parameters c and d. Aubresch then finds closing conditions on c and d (assuring that the solution W is in fact doubly periodic).

In 1989 Pinkhall and Sterling classified all CMC tori in [4]. Their idea is to write solutions to equation (8) as the flows of two commuting vector fields. Then one can integrate to get solutions and show that there exist only finitely many independent integrals. They then embed the ODE system in the Jacobian variety of the torus and find the closing conditions.

3.3 Kapouleas' Surfaces

In [16] Kapouleas produced many examples of noncompact embedded CMC surfaces. As a first step, he creates a central graph, consisting of vertices, edges, rays, and weights for each vertex. He requires that the edges of these graphs have lengths that are even integers and that the graphs are balanced around each vertex (see section 4.2). About each vertex he places a sphere of radius one. He places half a Delaunay surface about each ray, with necksize determined by the weight at the starting vertex of the ray. About the edges of length greater than 2, he places a piece of a Delaunay surface to connect the two spheres centered at the vertices which are the endpoints of the edge in question. Again, the necksize of this joining piece of Delaunay surface is determined by the weights of the vertices (which must be the same by balancing).

Next Kapouleas pieces all the surfaces together to form a smooth approximate solution. He pastes the spheres and pieces of Delaunay surfaces together with appropriately chosen cut-off functions. However, all the parts do not quite fit together without some sort of perturbation. For instance, the period of a Delaunay surface with small necksize is almost, but not quite, 2. So the Delaunay piece joining the two spheres mentioned above does not quite fit. To remedy this problem, Kapouleas first slightly perturbs the graph, and then slightly perturbs the necksizes of the Delaunay surfaces. After this step, he has a surface which has mean curvature one everywhere except for small bands near each neck of the Delaunay pieces. In these bands about the Delaunay necks the mean curvature is close to one.

Then Kapouleas solves the linearized problem (locally) on each bulge between the Delaunay necks. However, in these regions he must avoid the spherical harmonics which arise from eigenfunctions of the operator $\Delta + 2$ on $\mathbb{S}^2$. Thus he solves the linearized problem orthogonal to a finite dimensional "substitute kernel" on each bulge. A further difficulty in piecing together a global solution to the linearized operator from all these local solutions is that the global solution must be orthogonal to each of the substitute kernels mentioned above. This means that one must find a solution to the linearized problem which is orthogonal of to an infinite dimensional subspace. Finally, he must solve the nonlinear problem. To do this, Kapouleas shows one can find appropriate solutions for the linear problem after perturbing the graph mentioned above, and then uses a Leray-Schauder fixed point argument to show that a solution to the nonlinear problem for one of the perturbed graphs must exist.

4 General Properties of Almost Embedded CMC Surfaces

As stated above, we are concerned here with embedded CMC surfaces. However, many of the theorems still hold for a wider class of immersions, called *almost embeddings*.

Definition 1 *An immersion $X : \Sigma \to \mathbb{R}^3$ is called an almost embedding (or an Alexandrov embedding) if one can write Σ as the boundary of a solid handle-body Ω and X extends to be an immersion of Ω.*

One can think of this property as distinguishing an "outside" and an "inside" for the surface (the inside corresponding to the interior of the solid handle-body). Roughly speaking, the condition that a surface is almost embedded is the weakest condition one can place on the surface such that one can apply the Alexandrov reflection argument below.

4.1 Alexandrov Reflection

Alexandrov reflection is really an application of the maximum principle. To see how it works, we will first apply it to a compact CMC surface.

Let $X : \Sigma \to \mathbb{R}^3$ be a CMC embedding of a compact surface. Fix a large negative T so that the Σ lies completely above the plane $\pi = \{z = T\}$ (one can do this because Σ is compact). Let $\pi_t = \pi + (0, 0, t)$ be the translate of π by t in the z direction. Let Σ_t be the part of Σ which lies below π_t and let $\tilde{\Sigma}_t$ be the reflection of Σ_t through the plane π_t. For t small, both Σ_t and $\tilde{\Sigma}_t$ will be empty.

If t_0 is the first time of contact of π_t with Σ, then (locally) one can write Σ as a graph over π_{t_0}. Thus for $t = t_0 + \delta$, with $\delta > 0$ small, the reflected surface $\tilde{\Sigma}_t$ will lie completely inside Σ. In other words, for those values of t slightly larger that t_0, the reflected surface $\tilde{\Sigma}_t$ lies in the bounded component of $\mathbb{R}^3 \backslash \Sigma$. We pause to note that this is where we need Σ to be embedded.

Note that for t sufficiently large, Σ will lie completely below π_t (again, by the compactness of Σ), and so $\tilde{\Sigma}_t$ cannot be contained in the bounded component of $\mathbb{R}^3 \backslash \Sigma$ for all t. Let t_1 be the infimum of $t > t_0$ such that $\tilde{\Sigma}_t$ is not contained in the bounded component of $\mathbb{R}^3 \backslash \Sigma$ and let $\tilde{\Sigma}$ be the reflection of Σ through the plane π_{t_1}. Then in fact Σ and $\tilde{\Sigma}$ are tangent at some point p.

If the tangency at p is not a vertical tangency, write Σ and $\tilde{\Sigma}$ as graphs of u and u_1 (respectively) over the plane π_{t_1}. Let the tangency point p have coordinates (x, y) in this plane. Then $u(x, y) = u_1(x, y)$ and $\nabla u(x, y) = \nabla u_1(x, y)$. Also, u and u_1 both satisfy the same strongly elliptic equation (equation (1)). By the maximum principle, $u = u_1$, and therefore Σ locally agrees with $\tilde{\Sigma}$. Both surfaces are analytic and connected, so $\Sigma = \tilde{\Sigma}$.

If the tangency is at p is a vertical tangency, one needs to apply the Hopf boundary lemma (see Theorem 10 of Chapter 2 of [12]). In either case, we see that Σ has a plane of symmetry parallel to the $x - y$ plane. However, the $x - y$ plane had no special relation to the original surface Σ, and so we conclude that Σ has a plane of symmetry in every direction. Alexandrov used this to conclude

Theorem 3 (Alexandrov's Theorem): *Let $\Sigma \hookrightarrow \mathbb{R}^3$ be a compact embedded CMC surface. Then Σ is the round sphere.*

In the case where Σ is noncompact, a similar construction (found in [17]) still works. Let $\pi \subset \mathbb{R}^3$ be a plane with unit normal v. Let L be the line parameterized by $L(t) = tv$. For $t \in \mathbb{R}$ and $p \in \pi$ define

$$\pi_t = \pi + tv \qquad \Pi_t = \cup_{s \geq t} \pi_s \qquad L_p = L + p.$$

For any set $G \subset \mathbb{R}^3$ let

$$G_t = G \cap \Pi_t \qquad \tilde{G}_t = \{p + (t - r)v \mid p \in \pi, p + (t + r)v \in G_t\}.$$

Let Σ be an almost embedded surface, with $\Sigma = \partial\Omega$. First we restrict to a piece of Σ by taking an open set $W \subset \Omega$ and letting $S = \partial W \cap \Sigma$. Note that neither W nor S need be connected nor bounded. Suppose $p + tv \notin W$ for sufficiently large t. Let t_1 be the supremum of t such that $P + tv \in W$. Then $P_1 = p + t_1 v$ is the point of first contact of L_p with W. If this first contact is transverse, let t_2 be the supremum of $t < t_1$ such that $p + tv \notin W$. Then $P_2 = p + t_2 v$ is the point where L_p first leaves W. Otherwise, let $P_1 = P_2$. If P_1 and P_2 are both in S, then (as in [17]) we define

$$\alpha_1(p) = \frac{t_1 + t_2}{2}.$$

Notice that α_1 is not defined for all $p \in \pi$.

Lemma 4 (Korevaar, Kusner, Solomon): *Fix a plane π and its normal v. If, with $W \subset \Omega$ and $S \subset \Sigma$ as above, α_1 has a local interior maximum value z at $p \in \pi$ then the plane π_z is a plane of symmetry for Σ.*

Proof: First notice that $P_1(p)$ reflects to $P_2(p)$ through π_z, by construction. Pick a nearby q. Then by maximality $t_1(q) + t_2(q) \leq 2z$, and so

$$z - (t_1(q) - z) \geq t_2(q).$$

This means the reflection of $P_1(q)$ through π_z lies above $P_2(q)$. This implies a neighborhood of $P_2(p)$ in $\tilde{S}_z$ lies inside W. If $P_1(p) \neq P_2(p)$,Then S and $\tilde{S}_z$ are tangent at $P_2(p)$ with nonvertical tangent. If $P_1(p) = P_2(p)$, then S and $\tilde{S}_z$ are tangent with vertical tangent. In either case, argue as above using the maximum principle to see that π_z is a plane of symmetry for Σ. ■

The ends of Σ are the unbounded connected components of $\Sigma \backslash \mathbb{B}_r$ for sufficiently large r. Consider an end of Σ contained in a solid cylinder $C^+_{a,R}(P) = \{p + ta \mid |p - P| < R; \langle p - P, a\rangle = 0; t > 0\}$. We take $W = \Omega \cap C^+_{a,R}(P)$ and $S = \partial W \cap \Sigma$. Meeks proved in [6] that any end of a complete embedded CMC surface is contained in such a solid half-cylinder. Choose a plane π and normal v as above with $a \perp v$. Let $x(p) = \langle p, a\rangle$ and define

$$\alpha(x) = \max_{\langle p,a\rangle = x \geq 0} \alpha_1(p).$$

Then one can use this Alexandrov function and similar arguments as in the above Lemma to show:

Theorem 5 (Korevaar, Kusner, Solomon): *If Σ is a properly embedded CMC surface contained in a solid cylinder, then Σ has a rotational axis of symmetry parallel to the axis of the cylinder. Also, if Σ has finitely many ends and is contained in a half-space Π_0 for some plane π, then Σ has a plane of symmetry parallel to π and is thus contained in a solid slab.*

4.2 The Balancing Formula

CMC surfaces must also obey a balancing condition. This means that the ends of the surface Σ must be arranged to balance each other. To see this, we start with the following general proposition found in [8].

Theorem 6 (Kusner): *Let M be a 3 dimensional Riemannian manifold with $H_1(M)$ and $H_2(M)$ trivial. Let G be the isometry group of M and let $\mathfrak{g}$ be its Lie algebra. For some constant H, let Σ be a surface in M with mean curvature H. Then there is a natural cohomology class $\mu \in H^1(\Sigma) \otimes \mathfrak{g}^*$ defined as follows: let Γ be a 1-cycle in Σ with $\Delta \subset M$ such that $\partial\Delta = \Gamma$. Let ν be the oriented normal to Δ and η the oriented conormal to Γ. Let $Y \in \mathfrak{g}$. Then*

$$\langle \mu(\Gamma), Y \rangle = \int_\Gamma \langle \eta, Y \rangle - H \int_\Delta \langle \nu, Y \rangle.$$

The content of this theorem is that the formula above depends only on Y and the homology class of Γ. Let $\tilde{\Gamma}$ be another 1-cycle homologous to Γ in Σ. Because $H_1(M) = 0$ there are surfaces Δ and $\tilde{\Delta}$ in M with $\partial\Delta = \Gamma$ and $\partial\tilde{\Delta} = \tilde{\Gamma}$. Also, $\Gamma - \tilde{\Gamma}$ forms the boundary of some surface $S \subset \Sigma$. Then $\Delta - \tilde{\Delta} + S$ forms a 2-cycle in M. Because $H_2(M) = 0$, there is an open set $U \subset M$ such that $\partial U = \Delta - \tilde{\Delta} + S$. Now take $Y \in \mathfrak{g}$. Note $\phi_t = e^{tY}$ is a one-parameter family of isometries. In fact, the Killing field associated to ϕ_t is just the left-invariant vector field associated to Y. Therefore,

$$0 = \left.\frac{d}{dt}\right|_{t=0} [\mathrm{Area}(\partial(\phi_t(U))) - H\mathrm{Vol}(\phi_t(U))].$$

Applying Stokes' Theorem, the right hand side becomes

$$\int_\Gamma \langle \eta, Y \rangle - \int_{\tilde{\Gamma}} \langle \eta, Y \rangle - H \int_\Delta \langle \nu, Y \rangle + H \int_{\tilde{\Delta}} \langle \nu, Y \rangle,$$

which shows

$$\int_\Gamma \langle \eta, Y \rangle - H \int_\Delta \langle \nu, Y \rangle = \int_{\tilde{\Gamma}} \langle \eta, Y \rangle - H \int_{\tilde{\Delta}} \langle \nu, y \rangle.$$

Now consider the case $M = \mathbb{R}^3$ and take $Y = e_1, e_2, e_3$, the constant translational vector fields in the directions of the coordinate axes. Let $W \subset \Omega$ as above and $\partial W = S \cup Q$, where $S = \partial W \cap \Sigma$. Then the above theorem implies

$$\int_{\partial S} \eta - H \int_Q \nu = 0.$$

One useful choice of W is to take $W = \Omega \cap \mathbb{B}^3_R$ for some large R. Let $S = \partial W \cap \Sigma$. As mentioned above, Meeks proved in [6] that any end of Σ must be contained in a solid cylinder. So we can take R large enough so that ∂S is k disjoint simple closed curves, where Σ has k ends. Then we define the weight vector of an end as follows.

Definition 2 *For an end E which is contained in a solid half-cylinder $C^+_{a,r}(P)$, define the weight of the end E as*

$$w(E) = \int_{E\cap\pi} \eta - \int_{\pi\cap W} \nu$$

where $\pi = a^\perp$, arranged so that π intersects E transversally, ν is the normal to π, and η is the conormal to $\pi \cap E$.

By the balancing formula, the weights of all the ends of Σ must sum to the zero vector.

Consider the case of a Delaunay end. We can take $a = (0,0,1)$ and $E(t,\theta) = (\rho(t)\cos\theta, \rho(t)\sin\theta, t)$, and π any plane $\pi = \{z = z_0\}$. By symmetry, $w(E)$ must point along the z axis. Moreover, $\langle a, \nu\rangle = 1$ and $\langle a, \eta\rangle = (1+\rho_t^2)^{-\frac{1}{2}}$. Using $\mathrm{length}(\pi\cap E) = 2\pi\rho$ and $\mathrm{Area}(\pi \cap W) = \pi\rho^2$, we get

$$w = (\frac{2\pi\rho}{\sqrt{1+\rho_t^2}} - \pi\rho^2)(0,0,1).$$

One can check that

$$\frac{d}{dt}[\frac{2\pi\rho}{\sqrt{1+\rho_t^2}} - \pi\rho^2] = -\frac{\pi\rho\rho_t}{(1+\rho_t^2)^{\frac{3}{2}}}[\rho_{tt} - \frac{1}{\rho}(1+\rho_t^2) + 2(1+\rho_t^2)^{\frac{3}{2}}] = 0,$$

and so the coefficient of the above weight vector is a first integral of equation (9). In fact, if we normalize so that $\rho(0) = \epsilon$ is a minimum, then evaluating this constant at $t = 0$ shows $\frac{2\rho}{\sqrt{1+\rho_t^2}} - \rho^2 = 2\epsilon - \epsilon^2 = \tau$. Thus $\tau = 2\epsilon - \epsilon^2$ determines the weight of a Delaunay end of necksize ϵ.

4.3 Asymptotic Behavior of the Ends

Let E be a cylindrically bounded end of Σ, where Σ in a complete, noncompact CMC surface of finite topological type. Say E is contained in the half-cylinder $C^+_{a,R}(P)$. Let b, c be an orthonormal basis for $a^\perp$ and let $\omega(\theta) = b\cos\theta + c\sin\theta$. The result of [17] is that for t large enough, we can parameterize E as $\rho_E(t,\theta)\omega(\theta) + ta$.

Moreover, there exists an embedded Delaunay surface $D(t,\theta) = \rho_D(t)\omega(\theta) + ta$ such that for t_0 sufficiently large and some $\lambda > 0$ the following estimate holds:

$$\|\rho_E - \rho_D\|_{k,\alpha,(t_0-1,t_0+1)} = O(e^{-\lambda t_0}),$$

where $k \in \mathbb{N}$, $\alpha \in (0,1)$, and $\|\cdot\|_{k,\alpha,(t_0-1,t_0+1)}$ is the standard Hölder norm on $(t_0-1, t_0+1) \times \mathbb{S}^1$. In this sense, each end of a CMC surface Σ is asymptotic to a unique embedded Delaunay surface. The idea of the proof is to look at a slide-back sequence $E_k = E - t_k a$ and use a priori curvature estimates to extract a convergent subsequence. This shows compact subsets of E near infinity converge to translates of a fixed Delaunay surface. To eliminate this translation, one can write small translates of a Delaunay surface as normal variations of a fixed Delaunay surface, and then take a derivative of this family of Delaunay surfaces. These derivatives must have a certain form (see the next section), which the a priori estimates on curvature forbid.

5 The Jacobi Operator

The Jacobi operator of a immersed surface $X : \Sigma \to \mathbb{R}^3$ is the linearization of the mean curvature operator. More precisely, let ν be the normal to the surface $X(\Sigma)$ and define $X_t(p) = X(p) + tu(p)\nu(p) : \Sigma \to \mathbb{R}^3$ for some smooth function u. X_t is also an immersion for small t. Let H_t denote the mean curvature of the immersion X_t and let $H = H_0$. Then the Jacobi operator associated with the immersion X is the differential operator defined by

$$\mathcal{L}_X(u) = \left.\frac{dH_t}{dt}\right|_{t=0}.$$

Writing a Taylor expansion in t yields

$$H_t = H + t\mathcal{L}_X u + O(t^2).$$

One can show that

$$\mathcal{L}_X = \frac{1}{2}(\Delta_\Sigma + \|A_\Sigma\|^2)$$

where Δ_Σ is the Laplace-Beltrami operator of the metric induced on Σ by X and A_Σ is the second fundamental form of the immersion X (see Appendix C of [16]). Geometrically, one can think of solutions to the equation $\mathcal{L}_X u = 0$ as giving normal perturbations to X which preserve the mean curvature up to first order. In this way, the Jacobi operator is analogous to the well-known Jacobi equation for variations of a geodesic. Thus vector fields along $X(\Sigma)$ of the form $u\nu$ with $\mathcal{L}_X u = 0$ are called Jacobi fields. In somewhat of an abuse of notation, the functions u themselves are called Jacobi fields.

We will need to know about the mapping properties of $\mathcal{L}$. In particular, we will need to know on which function spaces $\mathcal{L}$ is Fredholm, injective, and surjective. Injectivity and surjectivity will stem in part from the following property.

Definition 3 *A complete CMC immersion $X : \Sigma \to \mathbb{R}^3$ of a noncompact surface with finite topology is said to be nondegenerate if the kernel $\mathcal{L}_\Sigma = \frac{1}{2}(\Delta_\Sigma + \|A_\Sigma\|^2)$ acting on L^2 is trivial.*

Roughly speaking, this property will allow us some control over the behavior of Jacobi fields (see Theorem 7).

In the following two subsections we will establish some technical results we will need to state and prove the conjectured result. In section 5.1 we will analyze the Jacobi operator on a Delaunay surface. In section 5.2 we will analyze the Jacobi operator on general k-ended CMC surfaces. We will pay particular attention to Jacobi fields which arise from the bottom of the spectrum of Δ_Σ.

5.1 The Jacobi Operator on a Delaunay Surface

Recall from section 3.1 that one can parameterize the embedded Delaunay surfaces as

$$D_\tau(s,\theta) = (\tau e^{\sigma(s)} \cos\theta, \tau e^{\sigma(s)} \sin\theta, k(s))$$

where ϵ is the necksize of the Delaunay surface D_τ, $\tau = 2\epsilon - \epsilon^2$,

$$\frac{d^2\sigma}{ds^2} + \frac{\tau^2}{2} \sinh 2\sigma = 0, \tag{10}$$

and

$$\frac{dk}{ds} = \frac{\tau^2}{2}(e^{2\sigma} + 1).$$

Given solutions σ and k to the above equations, the embedding D_τ is a conformal map with conformal factor $\tau e^{\sigma(s)}$ (see [18]). In these coordinates the Jacobi operator becomes

$$\mathcal{L}u = \frac{1}{\tau^2 e^{2\sigma}}(\partial_s^2 u + \partial_\theta^2 u + \tau^2 \cosh 2\sigma).$$

Thus solutions of the Jacobi equation solve the PDE

$$\partial_s^2 u + \partial_\theta^2 u + \tau^2 \cosh(2\sigma) u = 0. \tag{11}$$

We can separate variables and write

$$u(s,\theta) = \sum_{-\infty}^{\infty} \chi_j(\theta) u_j^{\pm}(s)$$

where $\partial_\theta^2 \chi_j = -j^2 \chi_j$ for $j \in \mathbb{Z}$, and

$$L_j(u_j^\pm) = \partial_s^2 u_j^\pm + \tau^2 \cosh(2\sigma) u_j^\pm - j^2 u_j^\pm = 0. \tag{12}$$

As we seek real solutions u, we will choose the eigenfunctions

$$\chi_j(\theta) = \begin{cases} \frac{1}{\sqrt{\pi}} \cos j\theta & \text{for} \quad j > 0 \\ \frac{1}{\sqrt{2\pi}} & \text{for} \quad j = 0 \\ \frac{1}{\sqrt{\pi}} \sin j\theta & \text{for} \quad j < 0 \end{cases} .$$

Note $\{\chi_j\}$ form an orthonormal basis for $L^2(\mathbb{S}^1)$ and so it is no loss of generality to write u as a sum this way. The functions $u_j^\pm$ are called the j-th eigenmodes of u.

One can identify the lower eigenmodes ($|j| \leq 1$) with explicit geometric deformations of D. First change variables in the above parameterization of D by letting $t = k(s)$ and $\rho(t) = \tau e^{\sigma(s)}$. Thus

$$D(t, \theta) = (\rho(t) \cos\theta, \rho(t) \sin\theta, t).$$

Note that in these coordinates the normal vector ν is given by $\nu(t, \theta) = \frac{1}{\sqrt{1+\rho_t^2}}(-\cos\theta, -\sin\theta, \rho_t)$. We wish to write a translation $D_\eta(t, \theta) = D(t, \theta) + (0, 0, \eta) = D(t', \theta') + u(t', \theta')\nu(t', \theta')$ as a normal variation of $D(t, \theta)$. We are left with three equations

$$\begin{cases} \rho(t)\cos\theta &= \rho(t')\cos\theta' - \frac{1}{\sqrt{1+\rho_t^2(t')}} u(t', \theta')\cos\theta' \\ \rho(t)\sin\theta &= \rho(t')\sin\theta' - \frac{1}{\sqrt{1+\rho_t^2(t')}} u(t', \theta')\sin\theta' \\ t + \eta &= t' + \frac{\rho_t(t')u(t', \theta')}{\sqrt{1+\rho_t^2(t')}} \end{cases} . \tag{13}$$

Squaring the first two equations of (13) and adding them together we get

$$\rho^2(t) = \rho^2(t') + \frac{u^2(t', \theta')}{1 + \rho_t^2(t')} - \frac{2\rho(t')u(t', \theta')}{\sqrt{1 + \rho_t^2(t')}}.$$

Notice that from this equation we can take u to be a function of t alone. Multiplying through by $1 + \rho_t^2(t')$ and rearranging yields

$$u^2(t') - 2\rho(t')\sqrt{1 + \rho_t^2(t')}u(t') + (\rho^2(t') - \rho^2(t))(1 + \rho_t^2(t')) = 0.$$

The quadratic formula then implies

$$u(t') = (\rho(t') - \rho(t))\sqrt{1 + \rho_t^2(t')}.$$

From the third equation of (13),

$$t - t' = \frac{u(t')\rho_t(t')}{\sqrt{1 + \rho_t^2(t')}} - \eta.$$

Thus

$$\rho(t) = \rho(t') + (t-t')\rho_t(t') + O(t-t')^2 = \rho(t') + \frac{\rho_t^2(t')u(t')}{\sqrt{1+\rho_t^2(t')}} - \eta\rho_t(t') + O(t-t')^2.$$

Using this expression for $\rho(t)$ yields

$$u(t') = \frac{\eta\rho_t(t')}{\sqrt{1+\rho_t^2(t')}} + O(t-t')^2$$

and thus the Jacobi field which generates this translational deformation of D is the function

$$u = u_0^+ = \frac{\rho_t}{\sqrt{1+\rho_t^2}} = \sigma_s. \tag{14}$$

Notice that $u(s) = \sigma_s(s)$ solves equation (12) for $j = 0$:

$$u_{ss} + \tau^2 u \cosh 2\sigma = \sigma_{sss} + \tau^2 \sigma_s \cosh 2\sigma = 0.$$

In fact, this equation is just the derivative of the equation (10). This computation shows that one can actually integrate one of the 0-mode Jacobi fields $\chi_0(\theta)u_0^+(s)$ and obtain a deformation of D given by translation along the axis of D.

Similarly, one can recover the two translations of the axis (from the 1 and -1 eigenmodes), and the two rotations of the axis (also from the 1 and -1 eigenmodes), and the family of surfaces one obtains by varying the necksize (the other of the 0 eigenmodes). To fix notation, we will always take u_j^+ to be the Jacobi fields which generate translations and u_j^- the Jacobi fields which generate either rotations of the axis of symmetry ($|j| = 1$) or variations in the necksize ($j = 0$). In particular, all the low eigenmodes $u_j^\pm(s)\chi_j(\theta)$ for $|j| \le 1$ grow at most linearly. Thus, $e^{-\delta|s|}u_j^\pm(s)\chi(\theta) \in L^2(D)$ for any $\delta > 0$. In fact, the low eigenmodes are the only Jacobi fields which are globally exponentially bounded. This motivates use of the following spaces.

Definition 4 *Given an immersion of k-ended surface $X : \Sigma \to \mathbb{R}^3$ where the ends can each be written as graphs over a cylinder, we say $u \in H^s_\delta(\Sigma)$ if upon restriction to each end $\mathcal{E}$, $e^{-\delta t}u \in H^s(\mathcal{E})$. Here we give the cylinder $(a,\infty) \times \mathbb{S}^1$ coordinates $t \in (a,\infty)$ and $\theta \in \mathbb{S}^1$.*

Notice that $e^{\epsilon t} \in H^s_\delta((0,\infty)\times\mathbb{S}^1)$ for all $\epsilon < \delta$, but not for $\epsilon \ge \delta$. Let $X : \Sigma \to \mathbb{R}^3$ be a complete *nondegenerate* immersion of a noncompact surface Σ of finite topology. For $\delta > 0$, $H^{s+2}_{-\delta}(\Sigma) \subset L^2(\Sigma)$, and so by the nondegeneracy assumption, $\mathcal{L}_\Sigma : H^{s+2}_{-\delta} \to H^s_{-\delta}$ is injective. Then by duality and the fact that $\mathcal{L}_\Sigma$ is self adjoint, $\mathcal{L}_\Sigma : H^{-s}_\delta \to H^{-s-2}_\delta$ is surjective. By elliptic regularity, $\mathcal{L}_\Sigma : H^{s+2}_\delta \to H^s_\delta$ is also surjective.

5.2 The Jacobi Operator on k-Ended Surfaces

Now consider a more general k-ended complete embedded CMC surface $X : \Sigma \to \mathbb{R}^3$. As noted above, each end of this surface is asymptotic to an embedded Delaunay surface D. Therefore, over this end, one can write the Jacobi operator as

$$\mathcal{L}_\Sigma = \mathcal{L}_D + e^{-\alpha s}\mathcal{R}$$

where s is the variable parameterizing the distance away from a compact set of a point on the end, α is a positive number, and $\mathcal{R}$ is a second order operator with smooth bounded coefficients. The deformations of the Delaunay surfaces corresponding to low eigenmode solutions found above are asymptotic Jacobi fields on Σ. Pick some large R such that $X(\Sigma)\backslash\mathbb{B}_R(0) = \cup_1^k E_j$ is a disjoint union of k ends, each of which is a graph of a function ρ_j over an embedded Delaunay surface D_j. Let χ be a smooth cut-off function with $\chi = 1$ on $\mathbb{B}_R(0)$ and $\chi = 0$ on $\mathbb{R}^3\backslash\mathbb{B}_{R+1}(0)$ and let $\tilde{\Sigma}$ be the surface which agrees with Σ inside $\mathbb{B}_R(0)$ and is the graph of $\chi \cdot \rho_j$ over each D_j outside $\mathbb{B}_R(0)$. Note Σ and $\tilde{\Sigma}$ are diffeomorphic and let $\Phi : \Sigma \to \tilde{\Sigma}$ be a diffeomorphism between them. In fact, one can choose the Φ to be the identity inside $\mathbb{B}_R(0)$. For $|j| \leq 1$, define

$$w_j^{\pm,i}(p) = \Phi^{-1}(\chi(p) \cdot u_j^{\pm,i}(p))$$

where $u_j^{\pm,i}$ is the Jacobi field corresponding to the jth eigenmode over the the Delaunay surface D_i. Notice that $\mathcal{L}_\Sigma w_i^{\pm,j}$ decays exponentially on each end of Σ. We will refer to $w_j^{\pm,i}$ as the asymptotic Jacobi fields arising from the jth eigenmode on E_i.

Definition 5 *Let $X : \Sigma \to \mathbb{R}^3$ be a complete noncompact CMC immersion. The deficiency space W is the span of all the asymptotic Jacobi fields arising from low eigenmode deformations of the underlying Delaunay ends; $W = \mathrm{span}\{w_i^{\pm,j} \mid 1 \leq j \leq k; i = -1, 0, 1\}$. The bounded null space B is the set of all Jacobi fields which do not grow exponentially, but also do not decay exponentially. In other words,*

$$B = \{u \mid \mathcal{L}_\Sigma u = 0; u \in H^s_\delta(\Sigma), u \notin H^s_{-\delta}(\Sigma)\ \forall\ \delta > 0\}.$$

These two spaces are related as follows (see the Linear Decomposition Lemmas of [20] and [22]):

Theorem 7 (Kusner, Mazzeo, Pollack) *Let $X : \Sigma \to \mathbb{R}^3$ be a complete CMC immersion of a noncompact surface with finite topology. Let $u \in H^{s+2}_\delta(\Sigma)$ and $f \in H^s_{-\delta}(\Sigma)$ for $\delta > 0$ and sufficiently small such that $\mathcal{L}_\Sigma u = f$. Then $u = w + \phi$ where $w \in W$ and $\phi \in H^{s+2}_{-\delta}(\Sigma)$.*

Thus there is a well defined map $\Pi : B \to W$ given by projection. Notice that if $u, v \in B$ and $\Pi(u) = \Pi(v) = w \in W$ then $\mathcal{L}_\Sigma(u - v) = 0$ and $u - v \in H^s_{-\delta}(\Sigma)$. If Σ is also nondegenerate, then $u = v$. Thus in the nondegenerate case this map $B \to W$ is injective. In this case, we will identify B with its image in W. For the general immersion (which may be degenerate) the element $\Pi(u) = w \in W$ determines $u \in B$ only up to terms which decay exponentially.

In fact, W and B carry more structure. To see this, first recall that given two solutions u_1 and u_2 to a linear second order ODE $u'' + pu' + qu = 0$, the Wronskian $\mathrm{Wr}(u_1, u_2) = u_1 u_2' - u_2 u_1'$ satisfies the equation $(\mathrm{Wr})' + p\mathrm{Wr} = 0$. Notice that equation (12) ($u_{ss} + \tau^2 u \cosh 2\sigma - j^2 u = 0$) is a linear second order ODE with no first order terms. So the Wronskian $\mathrm{Wr}(u_j^+, u_j^-) = u_j^+(s)\partial_s u_j^-(s) - u_j^-(s)\partial_s u_j^+(s)$ is a non-zero constant. We normalize $u_j^\pm$ such that $\mathrm{Wr}(u_j^+, u_j^-) = 1$.

Let W_j be the part of W arising from the j eigenmodes of the model Delaunay surfaces for the ends of Σ. Write $u, v \in W_0$ as

$$u = \sum_1^k (a_i u^i_{0,+} + b_i u^i_{0,-})$$

and

$$v = \sum_1^k (\alpha_i u^i_{0,+} + \beta_i u^i_{0,-})$$

where $u^i_{0,\pm}$ is the element of W arising from the $0, \pm$ eigenmode of the model Delaunay surface for the ith end. As in [22] we define

$$\omega(u, v) = \lim_{R\to\infty} \int_{\Sigma\cap\mathbb{B}_R(0)} (\mathcal{L}_\Sigma u)v - u(\mathcal{L}_\Sigma v) = \lim_{R\to\infty} \int_{\Sigma\cap\mathbb{B}_R(0)} (\Delta u)v - u(\Delta v)$$

where $\mathbb{B}_R(0)$ is a large ball as in the definition of W. Upon integrating by parts, we find

$$\begin{aligned}
\omega(u, v) &= \lim_{R\to\infty} \int_{\partial(\Sigma\cap\mathbb{B}_R(0))} \frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu} \\
&= \lim_{R\to\infty} [\sum_1^k [(a_i \beta_j - b_i \alpha_j)\mathrm{Wr}(u^i_{0,+}, u^i_{0,-}) \frac{1}{2\pi} \int_0^{2\pi} d\theta] + O(e^{-R})] \\
&= \sum_1^k (a_i \beta_i - b_i \alpha_i).
\end{aligned}$$

Thus ω is the standard symplectic structure on $\mathbb{R}^{2k}$. Similarly, W_1 and W_{-1} carry the standard symplectic structure on $\mathbb{R}^{2k}$ and so W carries the standard symplectic structure on $\mathbb{R}^{6k}$. From the definition of ω, $B \subset W$ is an isotropic subspace. By a relative index theorem (see [20]), $\dim B = 3k = \frac{1}{2}\dim W$ and thus $B \subset W$ is Lagrangian.

Given an end E, let $W_E = \mathrm{span}\{\Phi^{-1}(\chi(p)u_j^{\pm})\}$ where the $u_j^{\pm}$ are the low eigenmode Jacobi fields of the model Delaunay surface for E. Functions $u \in B$ such that $\Pi(u) \in W_E$ are Jacobi fields on Σ which decay exponentially on all but one end of Σ and grow at most linearly on the remaining end E. As remarked above, in the nondegenerate case we can identify B with a subspace of W. In this case we will again abuse notation slightly and say $u \in B \cap W_E$. Thus $u \in B \cap W_E$ corresponds to a deformation of Σ which fixes the asymptotics of all ends except E and changes the asymptotics of E. However, Σ and its deformations must balance in the following sense.

Lemma 8 *If $u \in B \cap W_E$ for some end E of a noncompact, complete, embedded CMC surface Σ of finite topology, then u can only correspond to an asymptotic translation of the end E. Notice that u decays exponentially on all ends but E because $u \in B \cap W_E$, and so a curve in moduli space with tangent vector u fixes the asymptotics of all ends except E.*

Proof: To each end E_j associate the vector $\tau_j \vec{a_j}$ where E_j is asymptotic to the Delaunay surface D_j with necksize ϵ_j, $\vec{a_j}$ is a unit vector parallel to the axis of symmetry of D_j and pointing in the direction in which E_j in unbounded, and $\tau_j = 2\epsilon_j - \epsilon_j^2$. Then it is a theorem of Kusner (see [8] or [17]) that

$$\sum_1^k \tau_j \vec{a_j} = 0.$$

Suppose $u \in B \cap W_E$ and relabel the ends such that $E = E_1$. Write

$$u(p) = \sum (a_j \Phi^{-1}(\chi(p)u_j^+(p)) + b_j \Phi^{-1}(\chi(p)u_j^-(p))).$$

Let Σ_t be a deformation of Σ such that

$$\left.\frac{d}{dt}\Sigma_t\right|_{t=0} = u\nu_\Sigma.$$

Then τ_1 and $\vec{a_1}$ could depend on t but τ_j and $\vec{a_j}$ are constant for $j \geq 2$. By balancing,

$$-\tau_1 \vec{a_1} = \sum_2^k \tau_j \vec{a_j}$$

and is thus constant. This implies that $b_j = 0$ for $j = -1, 0, 1$. Thus any $u \in B \cap W_E$ must correspond only to asymptotic translations of an the end E. ∎

5.3 An Application: Structure of the Moduli Space

One of the main goals of the theory behind CMC immersions is to understand their moduli spaces, as defined below.

Definition 6 *Fix natural numbers k and g. We denote by $\mathcal{M}_{g,k}$ the space of all CMC almost embeddings $X : \Sigma \to \mathbb{R}^3$ where Σ has genus g and k ends where immersions are identified if and only if they differ by a Euclidean motion. In the case where $g = 0$ we will write $\mathcal{M}_k$ instead of $\mathcal{M}_{0,k}$. We give these spaces the topology induced by the Hausdorff topology on the closed sets $\Sigma \cap \mathbb{B}_R$ for sufficiently large R.*

It is a theorem of Meeks (see [6] and [17]) that there are no one ended CMC surfaces. Another theorem of Korevaar, Kusner, and Solomon ([17]) states that any two ended CMC surface must be a Delaunay surface. Thus $\mathcal{M}_1 = \varnothing$ and $\mathcal{M}_2 = (0, 1]$ (the parameter being the necksize of the surface). More recently, Kusner, Grosse-Braukman and Sullivan have shown that $\mathcal{M}_3$ is homeomorphic to $\mathbb{B}^3$ (see [7]). The other moduli spaces are as yet unknown.

The general structure of the moduli spaces $\mathcal{M}_k$ also not completely understood. As mentioned in the definition, they have a natural topological structure, but in general they may not have a natural smooth structure. Below we will sketch the proof of the following theorem (Theorem 3.1 of [20]):

Theorem 9 (Kusner, Mazzeo, and Pollack) *Let $X : \Sigma \to \mathbb{R}^3$ be a complete, non-degenerate, CMC almost embedding of a k-ended surface, with $k > 2$. Then there exists an open neighborhood $\mathcal{U} \subset \mathcal{M}_k$ containing $X(\Sigma)$ which is the quotient of a real analytic $3k - 6$ dimensional manifold by a finite isotropy group.*

The general idea of the proof is to write CMC surfaces nearby $X(\Sigma)$ in moduli space as the zero-set of a function. The difficulty is that for some immersions $\tilde{X}$ which are "nearby" X cannot be written as normal variations of X. For instance, some functions in the bounded null space B may correspond to asymptotic Jacobi fields which rotate the axis of symmetry of the model Delaunay surface of some of the ends. Integrating these Jacobi field yields a one-parameter family of CMC immersions where some of the ends rotates as the parameter varies. However, even for small values of the parameter none of these surfaces can be written as a graph over the original immersion. To see this, think of rotating a cylinder perpendicular to its axis. No matter how small the angle of rotation, one cannot write the rotated cylinder as a graph over the original cylinder. Thus we need to consider Jacobi fields over surfaces obtained by deforming the original immersion by elements of its deficiency space.

Recall the construction of the deficiency space immediately preceding definition 5. The deficiency space W of the immersion X is a $6k$-dimensional vector space. Let $\tau \in \mathcal{W} \subset W$, where $\mathcal{W}$ is a small ball, and denote by X_τ the immersion one obtains via the above deformation. We write $X_{\phi,\tau}$ for immersions which of the form $X_{\phi,\tau}(p) = X_\tau(p) + \phi(p)\nu_\tau(p)$ where ν_τ is the unit normal vector to X_τ. For δ small and positive, let $\mathcal{V}$ be a small ball in $H^{s+2}_{-\delta}(\Sigma)$. Then CMC immersions nearby to X are zeroes of the function $N(\phi,\tau) = H(X_{\phi,\tau}) - 1$. Note that N is a real analytic function. Moreover, for $\eta \in W$ and $\phi \in H^{s+2}_{-\delta}(\Sigma)$, the directional derivative of N in the direction (ϕ,η) is $\mathcal{L}(\eta+\phi) = \frac{1}{2}(\Delta_X(\eta+\phi) + \|A_X\|^2(\eta+\phi))$. By the nondegeneracy of Σ, this is zero only if $\eta + \phi = 0$. So by the Implicit Function Theorem, the zero set $\mathcal{U}_0 = \{(\phi,\tau) \in \mathcal{V} \times \mathcal{W} \mid N(\phi,\tau) = 0\}$ is a real analytic manifold whose dimension that of the null space of $\mathcal{L}$ acting on $H^{s+2}_{-\delta} \times W$. This is the dimension of the bounded null space B, which is $3k$. The neighborhood of the moduli space $\mathcal{U}$ is the quotient of $\mathcal{U}_0$ by the Euclidean motions. In the case where Σ has at least three ends, the isotropy group $\mathrm{Iso}(\Sigma)$ is finite and possibly empty (this isotropy group is the groups of Euclidean motions which fix Σ). The dimension count of the neighborhood $\mathcal{U}$ is $3k - \dim(\mathrm{Isom}(\mathbb{R}^3)) + \dim(\mathrm{Iso}(\Sigma)) = 3k - 6$. Also note that $\mathcal{U}$ is smooth in the case where $\mathrm{Iso}(\Sigma) = \varnothing$. This is the case where the original surface $X(\Sigma)$ has no dihedral symmetries. In the case where Σ has two ends, Σ has a rotational symmetry. The isotropy group is generated by this rotation and hence one-dimensional. Therefore, the dimension of $\mathcal{M}_2$ is $3 \cdot 2 - 6 + 1 = 1$, which agrees with the theorem of Korevaar, Kusner, and Solomon that all two-ended CMC surfaces are Delaunay surfaces.

In general, the moduli space $\mathcal{M}_k$ has the structure of a real analytic variety with formal dimension $3k - 6$ for $k > 2$ (see theorem 4.1 of [20]).

6 Gluing

6.1 Other Gluing Constructions

The construction explained below most closely follows the gluing construction of Mazzeo, Pollack, and Uhlenbeck in [21]. In this paper they start with two compact Riemannian manifolds (M_1, g_1) and (M_2, g_2) of constant positive scalar curvature and construct a metric g of constant positive scalar curvature on the connect sum, provided the Jacobi operator on each original manifold was nondegenerate. The nonlinear operator they work with is the Yamabe operator, whose Jacobi operator (linearization) is $\Delta + n$. First they construct an approximate solution as follows. They remove small metric balls $B_{2\alpha_1}(p_1)$ and $B_{2\alpha_2}(p_2)$ from M_1 and M_2 respectively. For $0 < \epsilon < 1$ they identify the annuli $B_{\alpha_i}(p_i) \backslash B_{\epsilon\alpha_i}(p_i)$ by the rule $(r_1, \theta_1) \sim (r_2, \theta_2)$ if and only if $\theta_1 = \theta_2$ and $r_1 r_2 = \epsilon\alpha_1\alpha_2$, where (r_i, θ_i) are geodesic polar coordinates about p_i. On this manifold M_ϵ they construct a metric with a cut-off function. One can think of the gluing region (the identified annuli) as a neck which joins a punctured M_1 and a punctured M_2. Letting $\epsilon \to 0$ corresponds to making the joining neck long. Mazzeo, Pollack, and Uhlenbeck then prove that the approximate solution is nondegenerate for small ϵ and that the Greens kernel for the Jacobi operator is uniformly bounded. This allows them to iterate the Jacobi operator and the Greens kernel and find a solution. The analysis they perform differs in several key ways from the analysis in our case. First, only the 0 eigenmode solutions to this Jacobi operator have subexponential growth. Second, the equation they must solve is conformally invariant.

More recently, Mazzeo and Pacard ([18]) and Mazzeo, Pacard, and Pollack ([19]) have constructed CMC surfaces with another gluing method. Each construction finds a solution by solving an infinite dimensional family of boundary value problems, instead of constructing a approximate solution and perturbing. In [18], Mazzeo and Pacard glue half Delaunay surfaces to k-noids with catenoid ends. They show that, for small necksizes, one can match the Cauchy data and attach a Delaunay surface to a truncated end of a k-noid. In [19] Mazzeo, Pacard, and Pollack construct new CMC surfaces nearby $\Sigma_1 \# \Sigma_2$ where Σ_i are nondegenerate CMC surfaces, and one realizes $\Sigma_1 \# \Sigma_2$ by placing Σ_1 tangent to Σ_2 such that $\Sigma_1 \cap \Sigma_2$ is a isolated point near the point of tangency. For this construction they must match the Cauchy data on the boundaries of $\Sigma_1 \backslash \mathbb{B}_\epsilon(p_1)$, $\Sigma_2 \backslash \mathbb{B}_\epsilon(p_2)$, and a small catenoid neck.

6.2 Statement of Intended Results

The gluing construction described below is an "end-to-end" gluing. We construct an approximate solution (see section 6.3) by patching together some ends of CMC surfaces.

Conjecture 10 *Let Σ' be a nondegenerate three-ended CMC surface and pick an end E. Without loss of generality, align Σ' so that E has the x axis as its asymptotic axis of symmetry. Let the bounded null space B of Σ' satisfy one of the following conditions:*

1. $B \cap W_E = \varnothing$

2. $B \cap W_E = \operatorname{span}\{w_1^+\}$ *where* w_1^+ *is the asymptotic Jacobi field which corresponds to a translation of the axis of* E *in the* y *direction*

3. $B \cap W_E = \operatorname{span}\{w_{-1}^+\}$ *where* w_{-1}^+ *is the asymptotic Jacobi field which corresponds to a translation of the axis of* E *in the* z *direction.*

Let Σ'' *be the surface obtained from* Σ' *by rotating* Σ' *about an axis perpendicular to the axis on the model Delaunay surface for* E*, and possibly rotating about the axis to the model Delaunay surface for* E*. Let* $\bar{\Sigma}$ *be the surface one obtains by patching* Σ' *and* Σ'' *with a cut-off function along the end* E *(see section 6.3). Then there exists a 4-ended CMC surface* Σ *which is nearby* $\bar{\Sigma}$ *in the sense that the Hausdorff distance between* $\bar{\Sigma} \cap \mathbb{B}^3_R(0)$ *and* $\Sigma \cap \mathbb{B}^3_R(0)$ *is small. In addition, the asymptotic data for* Σ *is close to that of* $\bar{\Sigma}$*. Moreover, the new surface* Σ *is nondegenerate.*

Recall that functions in $B_{\Sigma'} \cap W_{E'}$ correspond to deformations of the surface Σ' which fix the asymptotics of all ends except E' and change the asymptotics of E'. The above conjecture is a special case of the following more general conjecture.

Conjecture 11 *Let* Σ_1 *and* Σ_2 *be complete nondegenerate embedded CMC surfaces of finite topology and* k_1 *and* k_2 *ends, respectively. Label an end* E_1 *of* Σ_1 *and an end* E_2 *of* Σ_2*. Suppose* E_1 *and* E_2 *are asymptotic to the same Delaunay surface. Without loss of generality, align* Σ_1 *and* Σ_2 *so that the* x *axis is the asymptotic axis of both* E_1 *and* E_2*. Moreover, suppose the bounded null space* B_1 *of* Σ_1 *and the bounded null space* B_2 *of* Σ_2 *both satisfy one of the following conditions:*

1. $(B_1 \cap W_{E_1}) \cap (B_2 \cap W_{E_2}) = \{0\}$

2. $B_i \cap W_{E_i} = \operatorname{span}\{w_1^+\}$ *where* w_1^+ *is the asymptotic Jacobi field which corresponds to a translation of the axis of* E_i *in the* y *direction for* both $i = 1$ *and* $i = 2$

3. $B_i \cap W_{E_i} = \operatorname{span}\{w_{-1}^+\}$ *where* w_{-1}^+ *is the asymptotic Jacobi field which corresponds to a translation of the axis of* E_i *in the* z *direction for* both $i = 1$ *and* $i = 2$

Let $\bar{\Sigma}$ *be the surface one obtains by patching* Σ_1 *and* Σ_2 *with a cut-off function along the ends* E_1 *and* E_2*, possibly after rotating* Σ_2 *about the common axis for* E_1 *and* E_2*. Then there exists a CMC surface* Σ *with* $k_1 + k_2 - 2$ *ends which is nearby* $\bar{\Sigma}$ *in the sense that the Hausdorff distance between* $\bar{\Sigma} \cap \mathbb{B}^3_R(0)$ *and* $\Sigma \cap \mathbb{B}^3_R(0)$ *is small. In addition, the asymptotic data for* Σ *is close to that of* $\bar{\Sigma}$*. Moreover, the new surface* Σ *is nondegenerate.*

This conjecture has much geometric appeal. By translating Σ_1 and Σ_2 along the x axis, one can arrange them so that the distance between Σ_1 and Σ_2 is exponentially small in a neighborhood of the gluing region (see section 6.3 for details). The difficulties lies in preventing Jacobi fields in B_{Σ_1} and B_{Σ_2} from combining to yield Jacobi fields on the approximate solution $\bar{\Sigma}$ with finite L^2 norm. See the first paragraph of section 6.6 for more details.

As mentioned in the introduction, the purpose of this project is to construct new examples of CMC surfaces. It grew out of an attempt to answer the following question of Kusner. Given a three-ended CMC surface Σ_1 with chosen end E_1, let Σ_2 be the surface one obtains by first reflecting Σ_1 through a plane perpendicular to the asymptotic axis of E_1 and then rotating by angle θ about that axis. Kusner asked the question of whether the resulting CMC surfaces one obtains by perturbing this approximate solution (which depends on the angle θ) forms a loop in moduli space. To answer this question one must first prove that one can find CMC surfaces by this gluing technique, and so we started the present investigations. The original question might now be rephrased as follows. First, can one then glue Σ_1 and Σ_2 together to produce a new CMC surface Σ? Second, as one varies θ through a full rotation, does one obtain a closed loop in moduli space?

Our approach for proving these conjectures is the following. First, we construct an approximate solution $\bar{\Sigma}$ from the original summands with a cut-off function (see section 6.3). One important property of the approximate solution is the mean curvature of the approximate solution is very close to 1 and the support of $H_{\bar{\Sigma}} - 1$ is a compact set. Next we study the linearized problem. In particular, we show that we can solve the equation

$$\mathcal{L}_{\bar{\Sigma}} u = f$$

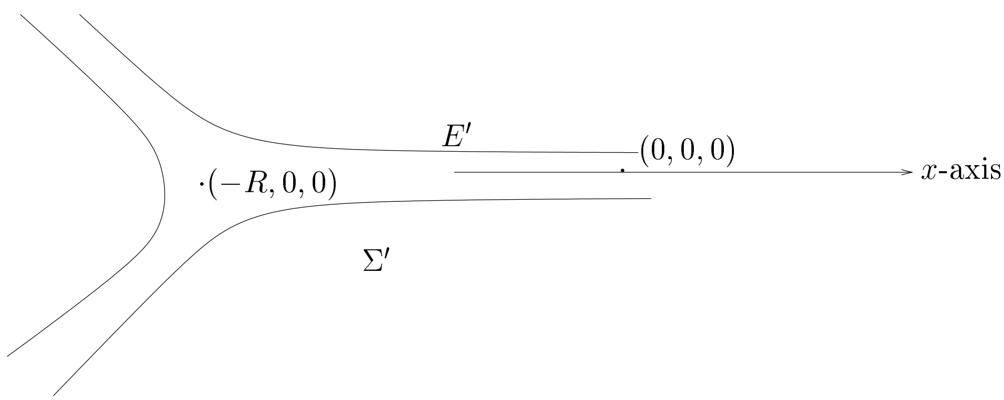
with $f \in H^s_{-\delta}$ and $u \in H^{s+2}_{\delta}$ for $\delta > 0$ small. To solve this equation we need to know that $\mathcal{L}_{\bar{\Sigma}}$ is injective on L^2 and that it is Fredholm for small positive weights. We also need a version of the earlier Linear Decomposition Lemma (theorem 7). The deficiency space for $\bar{\Sigma}$ should come from the ends of Σ' and Σ'' which are not used in the gluing construction. Finally, we need to know that $\mathcal{L}_{\bar{\Sigma}}$ composed with a Greens operator is a contraction. This will allow us to iterate the two operators and converge to a solution.

6.3 Constructing the Approximate Solution

The first step is to construct an approximate solution. We start with a surface $X' : \Sigma' \hookrightarrow \mathbb{R}^3$ which is a noncompact, complete embedded k-ended CMC surface of finite topology. The ends of the surface are the unbounded connected components of $\Sigma' \cap (\mathbb{R}^3 \backslash \mathbb{B}_{r_0})$ where r_0 is taken large enough so that the number of such components remains constant if r_0 increases. Roughly speaking, we can decompose the surface Σ' into a union of a compact piece and k noncompact ends. Label one of these ends E'. Recall the result of Korevaar, Kusner. and Solomon stated earlier, that the end E' is asymptotic to a Delaunay surface $D = D_\epsilon$ of necksize ϵ. In particular, if we write E' as the cylindrical graph of the function $\rho_{E'}$ and D as the cylindrical graph of the function ρ_D, the the result of Korevaar, Kusner, and Solomon implies the following estimate holds:

$$\|\rho_D(s) - \rho_{E'}(s,\theta)\|_{2,\alpha} = O(e^{-s_0})$$

for $0 < \alpha < 1$ and $s_0 \geq r+1$. The norm $\|\cdot\|_{2,\alpha}$ is the standard Hölder norm on $(s_0-1, s_0+1) \times \mathbb{S}^1$. Without loss of generality, we can suppose D has the x axis as its axis of symmetry and that ρ_D has a minimum occurring at $x = 0$. This amounts to a translation and rotation of Σ'. Moreover, by another translation of Σ' we can take the ball $\mathbb{B}_r$ in the above definition of the ends to be centered at $(-R, 0, 0)$ where R is a large positive parameter with $R > r+1$. After this second translation, we can write E' as a graph over the cylinder $(r-R, \infty) \times \mathbb{S}^1$.



Now note that under this situation $\|\rho_{E'}(s,\theta) - \rho_D(s)\|_{2,\alpha} = O(e^{-R})$ where the norm is the standard Hölder norm on bounded neighborhoods of $\{0\} \times \mathbb{S}^1$. We also remark that the embedding $X : \Sigma' \to \mathbb{R}^3$ depends on R. We will suppress this dependence, as two such embeddings of Σ' described above can only differ by a translation along the x axis.

At this point it is useful to discuss surfaces written as graphs over a cylinder $(a,b) \times \mathbb{S}^1$. Such surfaces can be parameterized as

$$(s,\theta) \mapsto (s, \rho(s,\theta)\cos\theta, \rho(s,\theta)\sin\theta)$$

for some positive function ρ. The induced metric from $\mathbb{R}^3$ is given by

$$g = (1 + (\partial_s\rho)^2)ds^2 + 2(\partial_s\rho)(\partial_\theta\rho)dsd\theta + \rho^2 d\theta^2.$$

They have a normal vector given by

$$\nu(s,\theta) = \frac{1}{\sqrt{\rho^2 + (\partial_\theta \rho)^2 + \rho^2(\partial_s \rho)^2}} (\rho \partial_s \rho, -\rho \cos\theta - (\partial_\theta \rho)\sin\theta, -\rho \sin\theta + (\partial_\theta \rho)\sin\theta).$$

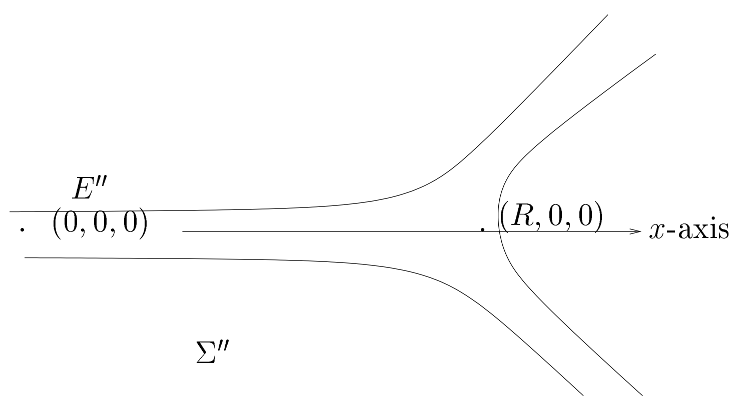
The have second fundamental form

$$A = \frac{[-\rho \partial_s^2 \rho ds^2 + 2((\partial_s \rho)(\partial_\theta \rho) - \rho(\partial_s \partial_\theta \rho))ds d\theta + (\rho(\rho - \partial_\theta^2 \rho) + 2(\partial_\theta \rho)^2)d\theta^2]}{\sqrt{\rho^2 + (\partial_\theta \rho)^2 + \rho^2(\partial_s \rho)^2}}.$$

In particular, we can read off from this information that the mean curvature is given by

$$H = \frac{-\rho^3(\partial_s^2 \rho - \partial_\theta^2 \rho) + \rho^2(1 + (\partial_s \rho)^2) + \rho((\partial_s \rho)(\partial_\theta \rho)(\partial_s \partial_\theta \rho) - \partial_\theta^2 \rho) + 2(\partial_\theta \rho)^2}{(\rho^2 + \rho^2(\partial_s \rho)^2 - 4(\partial_s \rho)^2(\partial_\theta \rho)^2)\sqrt{\rho^2 + (\partial_\theta \rho)^2 + \rho^2(\partial_s \rho)^2}}.$$

Let Σ'' be the image of Σ' under a rotation about the z axis by an angle of π



and let $\chi = \chi(s) \geq 0$ be a cutoff function where

$$\chi(s) = \begin{cases} 1 & \text{for} \quad s < -1 \\ 0 & \text{for} \quad s > 1 \end{cases}$$

and $\frac{d\chi}{ds}$, $\frac{d^2\chi}{ds^2}$ are bounded. Here we have to be careful about the parameterizations of E' and its image under the rotation E''. Both are asymptotic to the same Delaunay surface D, but they are asymptotic to opposite ends of D. We could just parameterize E'' by composing the rotation with the parameterization for E'. Then we have E' parameterized by

$$(s,\theta) \mapsto (-s, \rho_{E'}(-s,-\theta)\cos(-\theta), \rho_{E'}(-s,-\theta)\sin(-\theta)) : (r - R, \infty) \times \mathbb{S}^1.$$

But then the points $E'(s,\theta)$ and $E''(s,\theta)$ are far apart. Thus we need to adjust the parameterization to make $E'(s,\theta)$ close to $E''(s,\theta)$. We can do this by replacing s

with $-s$ and and θ with $-\theta$ after we rotate. Now we have E' written as the graph of $\rho_{E'}$ over the cylinder $(r-R,\infty)\times\mathbb{S}^1$ and E'' written as the graph of $\rho_{E''}$ over the cylinder $(-\infty,R-r)\times\mathbb{S}^1$. Moreover,

$$\|\rho_{E'}(s,\theta)-\rho_D(s)\|_{2,\alpha}=O(e^{-R})$$

for $s>r-R$ and

$$\|\rho_{E''}(s,\theta)-\rho_D(s)\|_{2,\alpha}=O(e^{-R})$$

for $s<R-r$. So

$$\|\rho_{E'}(s,\theta)-\rho_{E''}(s,\theta)\|_{2,\alpha}=O(e^{-R})$$

for $r-R<-1\le s\le 1<R-r$.

Now we construct the approximate solution $\bar{\Sigma}$ as follows. We can write part of $\bar{\Sigma}$ as a graph over the cylinder $(r-R,R-r)\times\mathbb{S}^1$. In the region corresponding to $r-R<s<-1$ let $\bar{\Sigma}$ be parameterized by

$$(s,\theta)\mapsto(s,\rho_{E'}(s,\theta)\cos\theta,\rho_{E'}(s,\theta)\sin\theta)$$

(i.e. in $(r-R,-1)\times\mathbb{S}^1$ $\bar{\Sigma}$ is the graph of $\rho_{E'}$). In the region $1<s<R-r$ let $\bar{\Sigma}$ be parameterized by

$$(s,\theta)\mapsto(s,\rho_{E''}(s,\theta)\cos\theta,\rho_{E''}(s,\theta)\sin\theta)$$

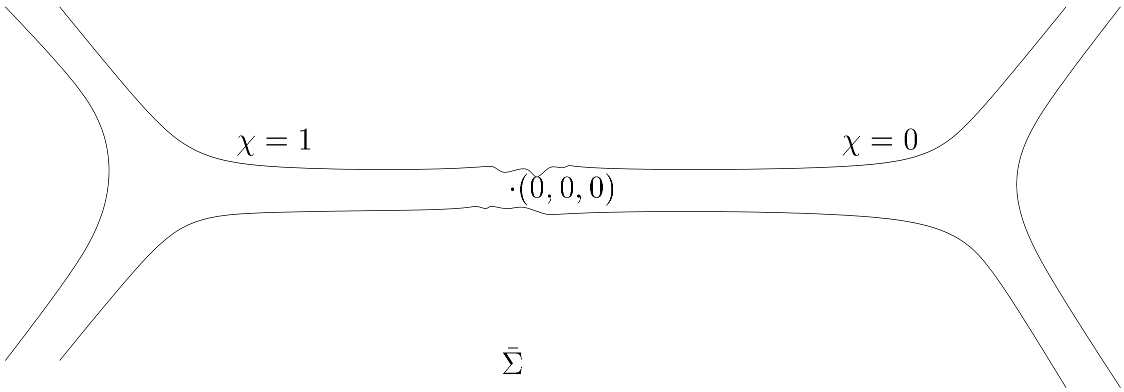
(i.e. $\bar{\Sigma}$ is the graph of $\rho_{E''}$ in $(r-R,-1)\times\mathbb{S}^1$). In the region $-1\le s\le 1$ parameterize $\bar{\Sigma}$ by

$$(s,\theta)\mapsto(s,\rho_{\bar{\Sigma}}(s,\theta)\cos\theta,\rho_{\bar{\Sigma}}\sin\theta)$$

where

$$\rho_{\bar{\Sigma}}(s,\theta)=\chi(s)\rho_{E'}(s,\theta)+(1-\chi(s))\rho_{E''}(s,\theta).$$

This gives a smooth surface with two boundary components written as a graph over a bounded cylinder. Because $\bar{\Sigma}$ and Σ' are given as graphs of the same function over the cylinder $(r-R,-1)\times\mathbb{S}^1$, we can extend $\bar{\Sigma}$ past the boundary component $\{r-R\}\times\mathbb{S}^1$ by letting it agree with Σ'. We can similarly extend $\bar{\Sigma}$ past the boundary component $\{R-r\}\times\mathbb{S}^1$ to agree with Σ''. Then $\bar{\Sigma}$ is a smooth surface, and it is CMC in the regions corresponding to $s<-1$ and $s>1$.



In the region $-1 \leq s \leq 1$ we can use the explicit formula for the mean curvature of a graph over a cylinder above and the fact that

$$\|\rho_{E'}(s,\theta) - \rho_{\bar{\Sigma}}(s,\theta)\|_{2,\alpha} = O(e^{-R})$$

to conclude that $H_{\bar{\Sigma}} = 1 + O(e^{-R})$.

In the gluing region $-1 \leq s \leq 1$ the mean curvature is $H_{\bar{\Sigma}} = \bar{H} = 1 - \psi$ where the error term $\psi = O(e^{-R})$ by the above computations. However, we can adjust this construction by changing the translation parameter R. In particular, we can make R as large as we please. Thus we can make this error ψ as small as we wish to start the construction.

6.4 Nondegeneracy of the Approximate Solution

In this gluing construction, we need the approximate solution $\bar{\Sigma}$ to be nondegenerate, at least when the summands Σ' and Σ'' are. Without nondegeneracy, we might not even be able to solve the linearized problem to find a perturbation of the approximate solution to a CMC surface. Unfortunately, this may not always be the case and we must place additional hypotheses on Σ' and Σ''.

Proposition 12 *We consider the situation as in section 6.3. Suppose that Σ' (and hence Σ'') is nondegenerate. Suppose further that $B_{\Sigma'}$ satisfies one of the following conditions:*

1. $B_{\Sigma'} \cap W_{E'} = \varnothing$
2. $B_{\Sigma'} \cap W_{E'} = \mathrm{span}\{w_1^+\}$ *where w_1^+ is the asymptotic Jacobi field which corresponds to a translation along the y axis.*

3. $B_{\Sigma'} \cap W_{E'} = \mathrm{span}\{w_{-1}^{+}\}$ *where* w_{-1}^{+} *is the asymptotic Jacobi field which corresponds to a translation along the* z *axis.*

Then, after possibly rotating Σ'' *about the* x*-axis by an arbitrarily small angle, for* R *sufficiently large* $\bar{\Sigma}$ *is nondegenerate.*

The proof of this proposition is somewhat complicated. The idea of the proof goes as follows. First divide $\bar{\Sigma}$ into three parts: the original two summands truncated at the end we are trying to glue, and the middle "neck" which joins them. We have a parameter R which controls how long the neck is. As $R \to \infty$, the three pieces converge to the original summands and the model Delaunay surface for the end in question.

Suppose the proposition were false. Then there would exist a sequence $R_i \to \infty$ and $0 \neq w_i \in L^2(\bar{\Sigma}_{R_i})$ such that $\mathcal{L}_i w_i = \mathcal{L}_{\bar{\Sigma}_{R_i}} w_i = 0$. We normalize the sequence so that $\|w_i\|_{L^2} = 1$. Define the surfaces $\Sigma_{1,R} = \bar{\Sigma} \cap \{x \leq -\frac{R}{2}\}$, $\Sigma_{2,R} = \bar{\Sigma} \cap \{x \geq \frac{R}{2}\}$, and $\Sigma_{3,R} = \bar{\Sigma} \cap \{-\frac{R}{2} \leq x \leq \frac{R}{2}\}$. Also define the surfaces $\tilde{\Sigma}_{1,R} = \bar{\Sigma} \cap \{x \leq -R\}$, $\tilde{\Sigma}_{2,R} = \bar{\Sigma} \cap \{x \geq R\}$, and $\tilde{\Sigma}_{3,R} = \bar{\Sigma} \cap \{-R \leq x \leq R\}$. Note that $\tilde{\Sigma}_{1,R_i}$ and $\tilde{\Sigma}_{1,R_j}$ differ only by a translation. We will sometimes find it convenient to ignore this difference. Similar remarks hold for $\tilde{\Sigma}_{2,R_i}$ and $\tilde{\Sigma}_{2,R_j}$.

Restricted to Σ_{1,R_i}, we can show that $w_i \to 0$ in C^k using an elliptic bootstrapping argument. Similarly, $w_i|_{\Sigma_{2,R_i}} \to 0$ in C^k and $w_i|_{\Sigma_{3,R_i}}$ converges to a Jacobi field in C^k. Now we have three cases to consider: $\|w_i\|_{L^2(\tilde{\Sigma}_{1,R_i})} \geq \delta > 0$, $\|w_i\|_{L^2(\tilde{\Sigma}_{2,R_i})} \geq \delta > 0$, or $\|w_i\|_{L^2(\tilde{\Sigma}_{1,R_i})} \to 0$ and $\|w_i\|_{L^2(\tilde{\Sigma}_{2,R_i})} \to 0$. This first two cases are similar; one can rule them out by the nondegeneracy of Σ' and Σ''. In the last case, we can take $\|w_i\|_{L^2(\tilde{\Sigma}_{3,R_i})} > \frac{1}{2}$. In this last case, we rescale w_i to get a sequence $\bar{w}_i$ whose weighted sup-norm is 1. We choose the weighting function to be 1 on $\tilde{\Sigma}_{1,R_i}$ and $\tilde{\Sigma}_{2,R_i}$ and large in the middle of $\tilde{\Sigma}_{3,R_i}$. Each weighted function $\bar{w}_i$ must attain its supremum. Again, we have three different cases: the supremum can occur near one of $\tilde{\Sigma}_{1,R_i}$ or $\tilde{\Sigma}_{2,R_i}$, in the middle of $\tilde{\Sigma}_3, R_i$, or at unbounded distance from $\tilde{\Sigma}_{1,R_i}$, $\tilde{\Sigma}_{2,R_i}$, and the middle of $\tilde{\Sigma}_{3,R_i}$. We can rule out the former case by nondegeneracy of Σ'' and Σ''. We can rule out the last case (unbounded distance from both $\tilde{\Sigma}_{1,R_i}$ and $\tilde{\Sigma}_{2,R_i}$ and the middle of $\tilde{\Sigma}_{3,R_i}$) by showing $\bar{w}_i$ must converge to a Jacobi field on a Delaunay surface with exponential decay on one end, where the coefficient in the exponent is too small. For the last case, we must bring the additional conditions on the bounded null space $B_{\Sigma'}$ to bear. If such a sequence $\bar{w}_i$ did exist, then we could produce a Jacobi field on either Σ' or Σ'' which is not allowed by the hypotheses of the Proposition. (See the opening paragraph of section 6.6 for remarks on what kind of behavior can occur if we do not make these assumptions.)

6.5 Linear Theory to be Proven

We wish to prove the following about the approximate solution.

First we wish to show that the operator $\mathcal{L}_{\bar{\Sigma}} : H^{s+2}_\delta \to H^s_\delta$ is Fredholm if and only if $\delta \notin \Gamma = \{\dots, -\gamma_2, -\gamma_1, 0, \gamma_1, \gamma_2, \dots\}$, where $0 < \gamma_j < \gamma_{j+1} \to \infty$. The idea behind this result is that Fredholm properties of $\mathcal{L}$ on H^{s+2}_δ can be understood in terms of the Fredholm properties of $\mathcal{L}$ on a compact piece K of Σ and on the ends E_i. In particular, if $\mathcal{L}$ is Fredholm on $H^{s+2}(K)$ and $H^{s+2}_\delta(E_i)$, then it is Fredholm on $H^{s+2}_\delta(\Sigma)$. Because $\mathcal{L}$ is elliptic, it is Fredholm on $H^{s+2}(K)$ by standard microlocal methods. Thus the question is reduced to finding out when the Delaunay Jacobi operator $\mathcal{L}_D$ is Fredholm on $H^{s+2}_\delta((0,\infty) \times \mathbb{S}^1)$. To address this problem, one introduces the Fourier-Laplace transform

$$\mathcal{F}(u)(\zeta, t, \theta) = \hat{u}(\zeta, t, \theta) = \sum_{\infty}^{\infty} e^{-i\zeta k} u(t + k, \theta).$$

Notice that if $u \in H^s_\delta((0,\infty) \times \mathbb{S}^1)$ then $\hat{u} \in \mathrm{Holo}((\Im(\zeta) < -\delta); H^2((0,\infty) \times \mathbb{S}^1))$. One conjugates $\mathcal{L}$ by $\mathcal{F}$ and multiplication by $e^{i\zeta t}$ to get an operator $\tilde{\mathcal{L}}(\zeta) : H^{s+2}((0,\infty) \times \mathbb{S}^1) \to H^s((0,\infty) \times \mathbb{S}^1)$ which depends holomorphically on ζ. The desired result follows from the analytic Fredholm theorem if we can show $\tilde{\mathcal{L}}$ is Fredholm for one value of ζ. The γ_j's arise as $-\Im(\zeta)$ where ζ is a pole of the Greens operator to $\tilde{\mathcal{L}}(\zeta)$.

Next we wish to show the deficiency space $W_{\bar{\Sigma}}$ is spanned by the asymptotic Jacobi fields on $\bar{\Sigma}$ which arise from ends of Σ' and/or Σ'' which are not E' or E''. Let u' be an asymptotic Jacobi field on Σ' which decays exponentially on E' and let u'' be an asymptotic Jacobi field on Σ'' which decays exponentially on E''. Then we can construct u on $\bar{\Sigma}$ which agrees with u' on $\Sigma' \backslash E'$ outside a compact set and agrees with u'' on $\Sigma'' \backslash E''$ outside a compact set via a cut-off function whose gradient is supported in the gluing region. This function u has the right asymptotics to begin the deficiency space of $\bar{\Sigma}$. By a dimension count, this construction would account for all of the deficiency space of $\bar{\Sigma}$.

Finally, we wish to show that if $u \in H^{s+2}_\delta$ and $f \in H^s_{-\delta}$ for $\delta > 0$ small and $\mathcal{L}_{\bar{\Sigma}} u = f$ then $u = w + \phi$ where $w \in W_{\bar{\Sigma}}$ and $\phi \in H^{s+2}_\delta$. This would follow from a contour integral. We can invert $\mathcal{F}$ by the following formula. Let $(t, \theta) \in (0,\infty) \times \mathbb{S}^1$ and let $t = l + \tilde{t}$ where $l \in \mathbb{Z}$ and $0 \leq \tilde{t} < 1$. Then

$$\mathcal{F}^{-1} u(t, \theta) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(\mu + i\nu) l} u(\mu + i\nu, \tilde{t}, \theta) d\mu.$$

Moreover, the above integral converges so long as $u(\mu + i\nu, \cdot, \cdot) \in H^s_{-\nu}$. Shifting ν amounts to shifting the contour in the integration of $\mathcal{F}^{-1}$. Shifting the contour across $\nu = 0$ in the definition of $\tilde{\mathcal{L}}$ above amounts to shifting across a pole of the Greens kernel of $\tilde{\mathcal{L}}$.

6.6 Questions

At first glance, the additional restrict on the form of $B_{\Sigma'}$ in Proposition 12 might seem unnecessary. However, the following can occur. Suppose Σ' has a Jacobi field which decays exponentially on all ends of Σ' except E' and is asymptotic on E' to the Jacobi field which translates the model Delaunay surface D along its axis. Then one can patch u on Σ' to $-u$ on Σ'' to get an approximate Jacobi field on $\bar{\Sigma}$ with finite L^2 norm. One can construct this approximate Jacobi field for all R in the construction of $\bar{\Sigma}$. Moreover, one can still construct this approximate Jacobi field after rotating Σ'' about the axis of D (recall the Jacobi fields on D which correspond to translations along the axis are rotationally invariant). This situation is similar to that of the Dirac operators studied by Cappell, Lee, and Miller in [1]. They prove that in the case corresponding to the one discussed in this paragraph, one can find arbitrarily small eigenvalues to the Dirac operator. One might hope to prove a similar theorem for the Jacobi operator.

The following is an interesting related question. When can one rule out this behavior in the bounded null space? In other words, can one characterize the CMC surfaces Σ with no Jacobi fields in B_Σ which decay exponentially on all but one end? As $\dim B = \frac{1}{2}\dim W$, one might expect $B \cap W_E = \{0\}$ most of the time. Another interesting question is the following: if the original summand Σ' has one of these troublesome Jacobi fields which prevents us from proving that the approximate solution is nondegenerate, when can one perturb Σ' in $\mathcal{M}_3$ and obtain a surface with which one can perform this gluing construction? Finally, one can pose the following question. For the above discussion, we have been concentrating on one chosen end E. If Σ does not satisfy the condition $B \cap W_E = \{0\}$, might it satisfy the same condition for another end $\tilde{E}$? For which nondegenerate CMC surfaces Σ can we *not* choose such an end? If $\Sigma \in \mathcal{M}_3$ is nondegenerate are does not satisfy the hypotheses of Proposition 12 for any choice of end E, then there must exist three curves in $\mathcal{M}_3$ through Σ which each correspond deforming the asymptotics of one end of Σ and leaving the asymptotics of the other ends fixed. Moreover, these curves intersect transversely at Σ. Recall that $\mathcal{M}_3$ is smooth near Σ, so it makes sense to speak of transverse intersections. If, further, we cannot perturb Σ in $\mathcal{M}_3$ so that we can perform the above gluing construction of some choice of end, then a small neighborhood of $\mathcal{M}_3$ must be foliated by the curves mentioned above. These three foliations would be transverse to each other. Can one ever rule out this behavior?

At this time, we cannot speculate about the answers to any of the above questions.

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