A Brief Review of Initial Data Engineering

Piotr T. Chruściel*
AEI, Golm† and LMPT, Tours

James Isenberg‡
University of Oregon

Daniel Pollack§
University of Washington

Abstract

We review the recently developed program for constructing and studying solutions of the Einstein constraint equations using gluing techniques. We discuss what we believe are sharp conditions sufficient for a pair of solutions to admit gluing via a connected sum or "wormhole", and describe how one carries out the gluing. We also discuss a number of useful applications.

1 Introduction

The initial value formulation is the most widely used procedure for constructing solutions of Einstein's gravitational filed equations, and the first step in carrying out such a construction is that of finding a set of initial data $(\Sigma^3, \gamma, K, \psi, \pi)$ which satisfies the Einstein constraint equations

$$R(\gamma) - |K|^2_\gamma - (\mathrm{tr}_\gamma K)^2 = 2\rho(\gamma, \psi, \pi) \tag{1.1}$$

$$D_i(K^{ij} - \mathrm{tr}_\gamma K \gamma^{ij}) = J(\gamma, \psi, \pi). \tag{1.2}$$

Here Σ^3 is a 3 dimensional manifold, γ is a Riemannian metric on Σ^3 with scalar curvature $R(\gamma)$, K is a symmetric tensor field, (ψ, π) represent any non gravitational fields which may be present, and ρ and J are the energy density and momentum density functions of these non gravitational fields.

*email piotr@gargan.math.univ-tours.fr, URL www.phys.univ-tours.fr/˜piotr

†Visiting fellow.

‡Partially supported by the NSF under Grant PHY-0099373; email jim@newton.uoregon.edu, URL www.physics.uoregon.edu/˜jim

§Partially supported by the NSF under Grant DMS-0305048; email pollack@math.washington.edu, URL: www.math.washington.edu/˜pollack

Since the early 1970's, the predominately used procedure for constructing and studying sets of data which satisfy the constraints has been the conformal method [3, 5] and the closely related conformal thin sandwich method [18]. These conformal techniques have been very useful both practically (for explicit construction of solutions) and theoretically (for proving theorems regarding properties of the solutions). They have not, however, been very successful in handling non constant mean curvature data sets. They are also somewhat limited in their ability to construct data for modeling prescribed physical systems.

As an alternative analytical tool, since 2000, "gluing techniques" have been developed for working with solutions of the constraints. One of the ideas in this context is to choose a pair of points p_1 and p_2 on a pair of known solutions $(\Sigma_1^3, \gamma_1, K_1, \psi_1, \pi_1)$ and $(\Sigma_2^3, \gamma_2, K_2, \psi_2, \pi_2)$ of the constraints, and construct a new solution on the connected sum manifold $\Sigma_1^3 \# \Sigma_2^3$ which agrees exactly with the original solutions outside a small neighborhood of the neck $S^2 \times I$ (for I an interval) now connecting the regions around the excised points p_1 and p_2.

Such a gluing operation cannot be carried out for *every* possible choice of pairs of data and pairs of points; however it can be done for generic such choices. We describe in Section 2 the explicit conditions on the data and the points which guarantee that gluing can be done, and we argue that these conditions are likely sharp. We then discuss in Section 3 some of the ideas and techniques used in carrying out the gluing and in showing that it can be carried out to completion in the appropriate cases. In Section 4 we review a number of the applications of gluing. Using it, we can attach black holes and wormholes to given spacetimes, we can prove that there are no topological restrictions on manifolds admitting asymptotically Euclidean or asymptotically hyperbolic solutions of the constraints, and we can show that there exist maximally developed vacuum solutions of Einstein's equations which contain a compact Cauchy surface but do not admit any constant mean curvature Cauchy surfaces. We make concluding remarks in Section 5.

Note that in this review paper, none of the proofs of the theorems we discuss are carried out in any detail; those appear in the succession of papers [7, 8, 13–15].

2 When Gluing Can be Done

Gluing techniques have been applied to solutions of a number of geometrically motivated PDE systems, so much so that they are now regarded as a standard tool in geometric analysis. In all cases, one has to impose a "non degeneracy" requirement which serves as a sufficient condition for successful gluing. This is true here as well for the Einstein constraint equations. While in our earlier results [14,15] we have stated alternatives, we now believe that the sharpest condition for gluing is based on the following

Definition 1 Let $(\Sigma^3, \gamma.K)$ be a set of initial data satisfying the Einstein vacuum constraint equations, and let $p \in \Sigma^3$ and let U be an open set containing p. The data has *No KIDs in U* if there do not exist non trivial solutions (N, Y) to the formal adjoint of the linearized constraint equations:

$$0 = \begin{pmatrix} 2(\nabla_{(i}Y_{j)} - \nabla^l Y_l g_{ij} - K_{ij}N + \mathrm{tr}K\, N g_{ij}) \\ \\ \nabla^l Y_l K_{ij} - 2K^l{}_{(i}\nabla_{j)}Y_l + K^q{}_l \nabla_q Y^l g_{ij} - \Delta N g_{ij} + \nabla_i \nabla_j N \\ +(\nabla^p K_{lp} g_{ij} - \nabla_l K_{ij})Y^l - N \mathrm{Ric}\,(g)_{ij} \\ +2NK^l{}_i K_{jl} - 2N(\mathrm{tr}\,K)K_{ij} \end{pmatrix}, \quad (2.1)$$

in U.

A similar definition holds for the Einstein-Maxwell, Einstein-Yang-Mills, Einstein-Vlasov, and other Einstein-matter systems. Geometrically, the No KIDs condition means that there are no Killing vectors defined on the domain of dependence of U in the spacetime development [17].

With the No KiDs definition in hand, we may now state the vacuum version of our main gluing result:

THEOREM 2.1 *Let $(\Sigma_1^3, \gamma_1, K_1)$ and $(\Sigma_2^3, \gamma_2, K_2)$ be a pair of smooth initial data sets which satisfy the vacuum ($\rho = 0$ and $J = 0$) constraint equations (1.1)-(1.2). Let $p_1 \in \Sigma_1^3$ and $p_2 \in \Sigma_2^3$ be a pair of points, with open neighborhoods $p_1 \in U_1$ and $p_2 \in U_2$ in which the No KIDs condition is satisfied. There exists a smooth data set $(\Sigma_1^3 \# \Sigma_2^3, \hat{\gamma}, \hat{K})$ which satisfies the Einstein constraint equations everywhere, and which agrees with (γ_1, K_1) and (γ_2, K_2) away from $U_1 \cup U_2$.*

To see that some non degeneracy condition (of the nature of *No KIDs*) is indeed needed for gluing to be permitted, it is useful to consider the following example: Let $(\Sigma_1^3, \gamma_1, K_1)$ be any solution of the constraints which has Σ_1^3 compact and γ_1 non flat, and let $(\Sigma_2^3, \gamma_2, K_2) = (R^3, flat, 0)$. If one could glue these two initial data sets, then one would have an asymptotically Euclidean data set which is not data for Minkowski spacetime, and yet is identical to such data outside of a compact region. It would follow that the data would have mass zero, which would be a violation of the positive mass theorem [20–22]. We are this forced to conclude that the gluing of these particular data sets cannot be done. We note that the data $(\Sigma_2^3, \gamma_2, K_2) = (R^3, flat, 0)$ violates the No KIDs condition at every point.

How restrictive is the No KIDs condition? As shown by Beig, Chrusciel and Schoen [4], a generic initial data set (appropriately defined) satisfies the No KIDs condition almost everywhere, so the condition is fairly mild.

While we believe that a result very similar to Theorem 2.1 holds for Einstein-Maxwell and other Einstein-matter field theories, no such theorem has yet been proven. We have in fact established conformal ("non localized") gluing results of the nature discussed in Section 3 for many Einstein-matter theories (See [13]); however to complete the job and obtain results ("localized") of the nature of Theorem 2.1, we need also to show that the Corvino-Schoen [6, 10, 11] type procedures can be extended to non vacuum theories. While this has not yet been done, there do not appear to be any fundamental impediments to doing it.

3 How Gluing Works

Our gluing results have been developed and proven in three stages (with Rafe Mazzeo contributing to the first two). The first result, appearing in [14], shows that if we have a pair of initial data sets $(\Sigma_1^3, \gamma_1, K_1)$ and $(\Sigma_2^3, \gamma_2, K_2)$ which both have constant mean curvature ("CMC") of the same value, and which satisfy the non degeneracy condition that, if either Σ_1^3 or Σ_2^3 is a closed manifold, then the corresponding K_1 or K_2 may not be identically zero, and the corresponding geometry γ_1 or γ_2 does not have a conformal Killing field with a zero at the gluing points p_1 or p_2, then a gluing of the following sort can be carried out (which we call a "non localized gluing"). One can find a one parameter family $(\Sigma_1^3 \# \Sigma_2^3, \gamma_T, K_T)$ of initial data sets, all of which satisfy the constraints everywhere on $\Sigma_1^3 \# \Sigma_2^3$, with (γ_T, K_T) approaching arbitrarily close to $(\gamma_1 K_1)$ and (γ_2, K_2), away from the "neck" (or bridge) connecting Σ_1^3 to Σ_2^3, as $T \to \infty$. Note that this type of "non localized" gluing, involving changes in the data everywhere, approaching the original data only in a limit, is the traditional form of gluing theorem which has generally been proven for geometric PDE systems such as those corresponding to constant scalar curvature metrics.

In our second work [15], we show that the CMC condition need only be imposed locally near the points about which one wishes to glue: gluing can then be carried out regardless of the mean curvature of the data sets away from the resulting neck, so long as an additional non degeneracy condition (concerning the linearization of the equations which arise in solving the constraints via the conformal method) holds.

Finally in our third work [7,8] we obtain the result stated in Theorem 2.1. In particular, we show that gluing can be carried out with no restrictions whatsoever on the mean curvature. In addition, we show that the gluing can be done in such a way that the data changes only locally; away from a neighborhood of the gluing points, the data remains completely unchanged. We call this "localized gluing".

Since the proof (and constructions) of our localized gluing result (Theorem 2.1) rely on those of the non localized result, and since the non localized gluing theorem is of interest in its own right, we now sketch the ideas used to obtain both results.

The proof of the non localized gluing theorem relies primarily on the conformal method. It proceeds roughly as follows: We first apply to each of the given sets of data $(\Sigma_1^3, \gamma_1, K_1)$ and $(\Sigma_2^3, \gamma_2, K_2)$ a conformal transformation which is singular at the gluing point, and the identity away from a neighborhood of that point. Along with the transformations $\gamma \to \gamma_c = \psi^4\gamma$ of the metrics , one transforms the traceless part σ of K via the formula $\sigma \to \sigma_c = \psi^{-2}\sigma$, thereby guaranteeing that if $div_\gamma \sigma = 0$, then $div_{\gamma_c}\sigma_c = 0$. As a result of these transformations, the data $(\Sigma_1^3, \gamma_1, K_1)$ and $(\Sigma_2^3, \gamma_2, K_2)$ near the gluing points are each replaced by data on an infinite half tube whose geometry approaches that of a round $S^2 \times R^1$ cylinder.

Next, we connect the two cylinders at a coordinate parameter length $T/2$ along each, using cutoff functions to smoothly join the data fields from each side. We obtain $(\Sigma_1^3 \# \Sigma_2^3, \hat{\gamma}_T, \hat{K}_T)$, which is no longer a solution of the constraints.

It follows from the construction of $\hat{\gamma}_T$ and $\hat{K}_T$ that $div_{\gamma_T}\sigma_T$ is non zero. The next step in the gluing construction is to find a vector field X_T such that $div_{\gamma_T}(LX_T) = -div_{\gamma_T}\sigma_T$, where L is the conformal Killing operator. For such a vector field, one verifies that $div_{\gamma_T}\tilde{\sigma}_T = 0$, where $\tilde{\sigma}_T := \sigma_T + LX_T$. In the course of finding X_T one also shows that, while LX_T is generally non zero everywhere on $\Sigma_1^3 \# \Sigma_2^3$, it approaches zero away from the bridge joining Σ_1^3 and Σ_2^3 for large T.

There is one remaining step to carry out in proving our first gluing result:

to solve the Lichnerowicz equation

$$\Delta_{\hat{\gamma}_T}\phi_T = \frac{1}{8}R_{\hat{\gamma}_T}\phi_T - \frac{1}{8}|\tilde{\sigma}|^2_{\hat{\gamma}_T}\phi^{-7} + \frac{1}{12}\tau^2\phi^5, \tag{3.1}$$

for the positive scalar ϕ_T, thus obtaining data $\bar{\gamma}_T = \phi_T^4\hat{\gamma}_T$ and $\bar{K}_T = \phi^{-2}\tilde{\sigma} + \frac{1}{3}\phi^4\hat{\gamma}_T\tau$ which satisfies the constraints everywhere on $\Sigma_1^3\#\Sigma_2^3$ for all T. (Here τ is the trace of K.) To prove that the solution ϕ_T exists, and further to prove that for large T, the data $(\bar{\gamma}, \bar{K})$ approaches the original data $(\gamma_1, K_1))$ and (γ_2, K_2) in appropriate regions, one needs to show that for an appropriate construction of a scalar ψ_T from the conformal blow up functions ψ_1 and ψ_2, together with cutoff functions, ψ_T is arbitrarily close to a solution of the Lichnerowicz equation for sufficiently large T.

We note that while this non localized gluing result is generally weaker than our later results, it does have the virtue that it allows Minkowski data to be glued (non locally) to other solutions of the constraints, since the hypotheses for our first result do not require a non degeneracy condition for data on R^3. There is no violation of the positive mass theorem, since after the non localized gluing is done, the data on $\Sigma^3\#R^3$ differs from Minkowski data in the asymptotically flat region, and in particular may have non zero mass.

As noted above, one of the key features of our work on the local gluing results of [7, 8] is the removal altogether of any CMC requirement on the sets of data to be glued. We do this through the use of a result due to Bartnik [2], which says that for any choice of a set of initial data satisfying the constraints, for any real number τ, and for any point p, there is always a deformation of the data in a neighborhood of p (via the Einstein evolution equations) which is still a solution of the constraint equations, and which has mean curvature equal to the constant value τ throughout that neighborhood. Using this result, we show that we may glue any given pair of sets of initial data satisfying the constraints in a fixed (now CMC) neighborhood of the points about which we wish to glue, regardless of their mean curvatures, since we may first deform the data to CMC data in neighborhoods of the gluing points, and then proceed as in our first result adapted to a manifold with boundary (as a boundary value problem).

The further changes involved in going from our global results in [14, 15] to Theorem 2.1 are two-fold. First, we replace our earlier non degeneracy condition by the No KIDs condition. Second, we introduce a non conformal deformation as a tool to replace non localized gluing by localized gluing, in which the data is completely unchanged away from the bridge connecting the gluing regions. Specifically, working with the constraints as an underdetermined system, we use techniques developed by Corvino and refined by Corvino-Schoen and Chruściel-Delay, to deform the data in an annular region around each end of the bridge in such a way that all of the gluing is done in the bridge and in its neighborhood, with no changes occurring away from this region. The details are found in [7, 8].

Our discussion here has focussed on gluing solutions of the Einstein vacuum constraint equations. As shown in [13], non localized conformal gluing can readily be carried out for the Einstein-Maxwell, Einstein-Yang-Mills, Einstein-fluid, Einstein-Vlaxov, and other non vacuum field theories. To obtain non localized gluing for these field equations, it will be necessary to first extend the Corvino-Schoen technique to these non vacuum theories; this has not been done yet. We do note however that we have, in [8], established a non-vacuum version of Theorem 2.1 which allows for arbitrary non-gravitational fields satisfying the dominant energy condition. These results insure that the dominant energy condition is preserved under the gluing; we do not, however, claim to control any additional evolution equations (such as the Maxwell equations) which these additional fields may satisfy.

4 Applications of Gluing

Studies of the gluing of solutions of the Einstein constraint equations have always been strongly motivated by applications. Indeed, we initiated the whole program with the goal in mind of constructing "skew data sets", which are instrumental in showing that there are vacuum maximal globally hyperbolic spacetime solutions of the Einstein field equations which have no CMC Cauchy surfaces. The idea is this: We define a *skew data set* to be a solution (Σ^3, γ, K) of the constraint equations with $\Sigma^3 = \Lambda^3 \# \Lambda^3$ being a manifold which does *not* admit a metric with scalar curvature $R \geq 0$ [19], and with $\eta : \Lambda^3 \# \Lambda^3 \to \Lambda^3 \# \Lambda^3$ a reflection map ($\eta^2 = Id$) such that $\eta^* \gamma = \gamma$ (reflection symmetry) and $\eta^* K = -K$ (reflection skew symmetry). As indicated by Eardley and Witt (unpublished) and Bartnik [1] the maximal spacetime development of a skew symmetric set of data *cannot* contain a CMC Cauchy surface. This is because one verifies that if a Cauchy surface with data $(\bar{\gamma}, \bar{K})$ is contained in the development of skew symmetric data, then there is a Cauchy surface with data $(\bar{\gamma}, -\bar{K})$ in the development as well. Thus if $(\bar{\gamma}, \bar{K})$ has CMC τ, then there is a Cauchy surface with CMC $-\tau$ as well. It then follows from barrier theorems [1], that if the development of a set of skew symmetric data has a CMC Cauchy surface, then it must have a maximal ($trK = 0$) Cauchy surface as well. Now if this were true, it would follow from the constraints that the data on this maximal Cauchy surface would have $R = |K|^2 \geq 0$. This would violate our assumption regarding the geometries admitted by Σ^3. We conclude that the development admits no CMC Cauchy surfaces.

To prove that there are vacuum spacetimes with no CMC Cauchy data surfaces, it remains to show that we can construct skew data sets. But this can be done readily via using our local gluing techniques as follows (see [8] for details). We first use the conformal method to find a solution of the constraints (T^3, γ, K) which has no KIDs. Then noting that if (T^3, γ, K) solves the constraints, then so to does $(T^3, \gamma, -K)$. We proceed to glue $(T^3, \gamma, -K)$ to (T^3, γ, K) at equivalent points. A bit of analysis shows that this gluing produces skew symmetric data, as desired.

Most of our applications of gluing are more direct than this one. We may, for example, use gluing to produce initial data for spacetimes containing multiple black holes. We do this by choosing an arbitrary set of asymptotically Euclidean initial data, choosing a set of points $\{p_1, p_2, ...p_N\}$ on that data set, and then gluing (non locally) a copy of Euclidean space data (with K=0) to each of the points $\{...p_k...\}$. (This can also be done locally with a generic asymptotically flat solution which satisfies the no KIDs condition on every open set.) Clearly this procedure produces data with N minimal surfaces, or apparent horizons. As shown in [9], in fact, in certain situations, one can verify that N independent black holes develop.

We can also use gluing to add an arbitrary number of wormholes, at least for a short period, to a given spacetime. Indeed, given a set of constraint-satisfying initial data (Σ^3, γ, K) and a choice of a pair of open regions U and W in Σ^3, we can use gluing to find a new solution which is identical to (Σ^3, γ, K) outside of U and W, and which contains an arbitrary number of wormholes connecting U and W. To do this, we simply note that with small deformations, we can guarantee that U and W admit no KIDs; further, we note that while our gluing results have been stated for points on independent sets of data, in fact one readily shows using the same techniques that gluing can be carried out for two points on the same data set (see [14]). This tells us nothing about the long time future development of an initial data set with multiple wormholes.

We note one further application: verifying that there are no topological restrictions on constraint-satisfying initial data sets. To show this, we first recall that since any closed three manifold Σ^3 admits a constant negative scalar curvature metric $\hat{\gamma}$, one can always produce constraint-satisfying data on Σ^3 simply by choosing K to be pure trace of the right magnitude. This has long been known. Our new application is to show that for any closed three manifold Σ^3 we can always find asymptotically Euclidean as well as asymptotically hyperbolic solutions of the constraints on Σ^3 with a point removed. One finds these by gluing either an asymptotically flat or asymptotically hyperbolic solution of the constraints on R^3 to a small deformation of the simple solution $(\Sigma^3, \gamma, K = c\gamma)$. For the details of these applications we refer the interested reader to [14] (for the asymptotically hyperbolic case) and [15] (for the asymptotically Euclidean case).

5 Concluding Remarks

It is unlikely that the gluing results we have obtained for solutions of the Einstein constraint equations can be significantly strengthened, apart from allowing the presence of non gravitational fields (with coupled, additional evolution equations). (Note that while we have not discussed the issue here, as shown in our papers, all of our results do hold for general dimension.)

On the other hand, we believe that there might be a way to generalize gluing in the following sense: One might consider gluing along corresponding embedded submanifolds, of codimension at least three, of the initial data sets, rather than at corresponding points. Recent work of Mazzieri [16] with constant scalar curvature metrics suggests that this should indeed work. If so, gluing could prove useful in the study of the stability of black rings, and more generally, the topology of all "black objects" in higher dimensional spacetimes.

Whether or not this new version of gluing works, it is clear that gluing provides a powerful new tool for constructing and studying solutions of the Einstein constraint equations.

References

[1] R. Bartnik, *Remarks on cosmological spacetimes and constant mean curvature surfaces,* Comm. Math. Phys. **117** (1988) no. 4, 615-624.

[2] R. Bartnik, *Regularity of variational maximal surfaces*, Acta Math. **161** (1988), 145–181.

[3] R. Bartnik and J. Isenberg, *The constraint equations*, 50 years of the Cauchy problem in general relativity (P.T. Chruściel and H. Friedrich, eds.), Birkhaeuser, Basel, 2004, gr-qc/0312047.

[4] R. Beig, P.T. Chruściel, and R. Schoen, *KIDs are non-generic*, Ann. Henri Poincaré **6** (2005), no. 1, 155–194; gr-qc/0403042.

[5] Y. Choquet-Bruhat and J. York, *The Cauchy Problem,* in General Relativity, A. Held ed., Plenum 1979.

[6] P.T. Chruściel and E. Delay, *On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications*, Mém. Soc. Math. de France. **94** (2003) 1–103. gr-qc/0301073.

[7] P. T. Chruściel, J. Isenberg and D. Pollack, *Gluing initial data sets for general relativity,* Phys. Rev. Lett. **93**, 081101 (2004); ArXiv: gr-qc/0409047

[8] P. T. Chruściel, J. Isenberg and D. Pollack, *Initial data engineering,* Comm. Math. Phys. **257** (2005) 1, 29–42; ArXiv: gr-qc/0403066.

[9] P. T. Chruściel and R. Mazzeo, *On "many black hole" vacuum spacetimes*, Class. Quantum Grav. **20** (2003), 1–26.

[10] J. Corvino, *Scalar curvature deformation and a gluing construction for the Einstein constraint equations,* Comm. Math. Phys. **214** (2000), no. 1, 137–189.

[11] J. Corvino and R. Schoen, *On the asymptotics for the vacuum Einstein constraint equations*, J. Differential Geometry **73** (2006) 185–217; gr-qc/0301071.

[12] J. Isenberg, *Constant mean curvature solutions of the Einstein constraint equations on closed manifolds,* Class. Quantum Grav. **12** (1995) 2249–2274.

[13] J. Isenberg, D. Maxwell and D. Pollack, *A gluing constructions for non-vacuum solutions of the Einstein constraint equations,* ArXiv: gr-qc/0501083 Advances in Theoretical and Mathematical Physics, Vol. **9** No. 1, (2005) 129–172.

[14] J. Isenberg, R. Mazzeo and D. Pollack, *Gluing and Wormholes for the Einstein constraint equations,* Comm. Math. Phys. **231** (2002) 3, 529–568; ArXiv: gr-qc/0109045.

[15] J. Isenberg, R. Mazzeo and D. Pollack, *On the topology of vacuum space-times,* Ann. Henri Poincaré **4** (2003) 2, 369–383; ArXiv: gr-qc/0206034.

[16] L. Mazzieri, *Generalized connected sum construction for constant scalar curvature metrics*, math.DG/0602471

[17] V. Moncrief, *Spacetime symmetries and linearization stability of the Einstein equations I*, J. Math. Phys. **16** (1975), 493–498.

[18] H. Pfeiffer and J. York , *Extrinsic curvature and the Einstein constraints*, Phys. Rev. D **67** (2003) 094022

[19] R. Schoen and S.T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), no. 1-3, 159–183.

[20] R. Schoen and S.T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76.

[21] R. Schoen and S.T. Yau, *Proof of the positive mass theorem. II*, Comm. Math. Phys. **79** (1981), no. 2, 231–260.

[22] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402.