# Complexity for Modules over the Lie Superalgebra $\mathfrak{gl}(m|n)$

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# References

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# Lie Superalgebras

Throughout let $k = \mathbb{C}$. Let $\mathfrak{g}$ be a *Lie superalgebra* which is a $\mathbb{Z}_2$-graded vector space

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

with a bracket operation $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which preserves the $\mathbb{Z}_2$-grading and satisfies graded versions of the usual Lie bracket axioms.

## Definition

A finite dimensional Lie superalgebra $\mathfrak{g}$ is called *classical* if there is a connected reductive algebraic group $G_{\bar{0}}$ such that $\mathrm{Lie}(G_{\bar{0}}) = \mathfrak{g}_{\bar{0}}$ and an action of $G_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ which differentiates to the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$.

# $\mathfrak{g}l(m|n)$

## Example

The underlying vector space for $\mathfrak{g} = \mathfrak{gl}(m|n)$ is the set of $(m+n) \times (m+n)$ matrices over $\mathbb{C}$. We have $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, where $\mathfrak{g}_{\bar{0}}$ consists of matrices of the form:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Moreover, $\mathfrak{g}_{\bar{1}}$ consists of matrices

$$\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}.$$

The supercommutator is given by

$$[E_{i,j}, E_{k,l}] = E_{i,j}E_{k,l} - (-1)^{\bar{E}_{i,j}\bar{E}_{k,l}} E_{k,l}E_{i,j}.$$

# A Natural Self-Injective Category

Consider the category $\mathcal{F} = \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$ of finite-dimensional modules for a classical Lie superalgebra (i.e., $\mathfrak{g} = \mathfrak{g}l(m|n)$) which are completely reducible over $\mathfrak{g}_{\bar{0}}$ then

• this is a highest weight category (as observed by Brundan),

• there are *infinitely many simple modules*,

• $\mathcal{F}$ is self-injective (i.e., projective is equivalent to injective) [BKN3],

• projective resolutions have infinite length, and the terms can grow in dimension.

Thus, the cohomology can also grow in dimension so one is motivated to study these objects with ideas and tools from modular representation theory.

# Complexity

## Definition

Let $\mathcal{V} = \{V_t : \ t \in \mathbb{N}\} = \{V_\bullet\}$ be a sequence of finite dimensional $\mathbb{C}$-vector spaces. The *rate of growth* of $\mathcal{V}$, $r(\mathcal{V})$, is the smallest positive integer $c$ such that $\dim_k V_t \leq C \cdot t^{c-1}$ for some constant $C > 0$. If no such integer exists then $\mathcal{V}$ has infinite rate of growth.

## Definition

Let $M \in \mathcal{F}$ and $P_\bullet \twoheadrightarrow M$ be a minimal projective resolution for $M$. Following Alperin (1977), we define the *complexity of $M$* to be $r(\{P_n : \ n = 0, 1, 2, \dots\})$.

## Theorem (BKN3)

*Let $M$ be an object of $\mathcal{F}$. Then*

(i) $c_{\mathcal{F}}(M) = 0$ *if and only if $M$ is projective;*

(ii) $c_{\mathcal{F}}(M) \leq \dim \mathfrak{g}_{\bar{1}}$.

## Theorem (BKN3)

*Let $M \in \mathcal{F}$ and let $P_{\bullet} \twoheadrightarrow M$ be a minimal projective resolution. Then*

$$c_{\mathcal{F}}(M) := r(P_{\bullet}) = r\left(\mathrm{Ext}^{\bullet}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M, \bigoplus S^{\dim P(S)})\right)$$

*where the sum is over all simple modules $S$ in $\mathcal{F}$, and $P(S)$ is the projective cover of $S$.*

# Complexity for $\mathfrak{gl}(1|1)$

## Example

Let $\mathfrak{g} = \mathfrak{gl}(1|1)$. The simple modules in the principal block are one dimensional and indexed by $L(\lambda \,|\, -\lambda)$ where $\lambda \in \mathbb{Z}$. The projective cover $P(\lambda \,|\, -\lambda)$ of $L(\lambda \,|\, -\lambda)$ is four dimensional.

The minimal projective resolution of the trivial module $L(0 \,|\, 0)$ is given by

$$\cdots \to P(1 \,|\, -1) \oplus P(-1 \,|\, 1) \to P(0 \,|\, 0) \to L(0 \,|\, 0) \to 0.$$

Therefore, $\dim P_n = 4(n+1)$ and $c_{\mathcal{F}}(L(0 \,|\, 0)) = 2$ (rate of growth of the minimal proj. resolution).

In fact, one can easily show that $c_{\mathcal{F}}(L(\lambda \,|\, -\lambda)) = 2$ for all $\lambda \in \mathbb{Z}$. The atypicality of every simple module in the prinicipal block is one and equal to $\dim \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{g}_{\bar{0}}; \mathbb{C})$.

# Kac and Simple Modules for $\mathfrak{gl}(m|n)$

We have a triangular decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{p}^{\pm} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\pm}$ (Type I Lie super algebra). Let $X^+$ denote the parameterizing set of highest weights for the simple finite dimensional $\mathfrak{g}_0$-modules. For $\lambda \in X^+$, let $L_0(\lambda)$ denote the simple $\mathfrak{g}_0$-module of highest weight $\lambda$. View $L_0(\lambda)$ as a simple $\mathfrak{p}^{\pm}$–supermodule via inflation. Set

$$K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} L_0(\lambda) \qquad \text{and} \qquad K^-(\lambda) = \operatorname{Hom}_{U(\mathfrak{p}^-)}(U(\mathfrak{g}), L_0(\lambda))$$

be the *Kac module* and the *dual Kac module*, respectively

The simple modules in $\mathcal{F}$ are paramertized by $X^+$ and can be realized as the quotients of the Kac modules. The simple modules will be denoted by $L(\lambda)$, $\lambda \in X^+$.

# Support Variety Theories

There are three "support variety" theories which will be relevant for our work. Each of these satisfies the property of support datum as defined by Balmer.

- Varieties for $\mathfrak{g}_{\pm 1}$
- Duflo-Serganova Varieties
- Varieties arising from $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C})$ ([BKN1], [BKN2])

[Compare with the Chevalley Restriction Theorem for complex semisimple Lie algebras: $\mathcal{N}$ nilpotent cone, $S(\mathfrak{g}^*)^G \to S(\mathfrak{h}^*)^W$].

# Varieties for $\mathfrak{g}_{\pm 1}$

Observe that $\mathfrak{g}_{\pm 1}$ is an abelian Lie superalgebra, thus

$$R = \mathrm{H}^\bullet(\mathfrak{g}_{\pm 1}, \mathbb{C}) = \mathrm{H}^\bullet(\mathfrak{g}_{\pm 1}, \{0\}, \mathbb{C}) \cong S(\mathfrak{g}_{\pm 1}^*)$$

Let $\mathcal{C}$ be the category of finite dimensional $\mathfrak{g}_{\pm 1}$-supermodules. If $M \in \mathcal{C}$, then one can define the $\mathfrak{g}_{\pm 1}$ *support variety* of $M$. Set

$$I_M = \{r \in R \mid r.m = 0 \text{ for all } m \in \mathrm{Ext}^\bullet_{\mathcal{F}}(M, M)\}$$

and then the support variety of $M$ is

$$\begin{aligned} \mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) &= \mathrm{MaxSpec}\,(R/I_M) \\ &\cong \{x \in \mathfrak{g}_{\pm 1} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}. \end{aligned}$$

Here $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$ is canonically isomorphic to the "rank variety", and $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$ detects $\mathfrak{g}_{\pm 1}$ projectivity.

# Connections with the Complexity over $(\mathfrak{p}^{\pm}, \mathfrak{g}_{\bar{0}})$

## Theorem (BKN4)

*Let $\mathfrak{g}$ be a Type I classical Lie superalgebra and let $M$ be a module in $\mathcal{F} = \mathcal{F}_{(\mathfrak{p}^{\pm}, \mathfrak{g}_0)}$. Then*

$$c_{\mathcal{F}}(M) = \dim \mathcal{V}_{\mathfrak{g}_{\pm 1}}(M) = \dim \mathcal{V}^{\text{rank}}_{\mathfrak{g}_{\pm 1}}(M).$$

Since all modules in $\mathcal{F}$ are $G_{\bar{0}}$-modules there are only finitely many possibilities for $\mathcal{V}_{\mathfrak{g}_{\pm 1}}(M)$. Note $G_{\bar{0}}$ acts on the variety $\mathfrak{g}_1$ via the adjoint action (i.e., $(A, B) \cdot x = AxB^{-1}$ for $A \in GL(m)$, $B \in GL(n)$, $x \in \mathfrak{g}_1$).

The orbits are

$$(\mathfrak{g}_1)_r = \{\, x \in \mathfrak{g}_1 \mid \operatorname{rank}(x) = r \,\}$$

for $0 \leq r \leq \min(m, n)$. In particular, we have

$$(\mathfrak{g}_1)_r = G_0.x_r,$$

where $x_r$ is any fixed matrix of rank $r$. The closure of $(\mathfrak{g}_1)_r$ is

$$\overline{(\mathfrak{g}_1)_r} = \{\, x \in \mathfrak{g}_1 \mid \operatorname{rank}(x) \leq r \,\};$$

thus $\overline{(\mathfrak{g}_1)_r} \subset \overline{(\mathfrak{g}_1)_s}$ if and only if $r \leq s$. Hence, the graph (Hasse diagram) which describes the partial ordering given by inclusion of orbit closures is a simple chain.

# Duflo-Serganova Varieties

Consider the subvariety of $\mathfrak{g}_{\bar{1}}$ given by

$$\mathcal{X} = \left\{ x \in \mathfrak{g}_{\bar{1}} \mid x^2 = [x, x]/2 = 0 \right\}.$$

For $M \in \mathcal{F}$, Duflo and Serganova defined the *associated variety*

$$\begin{aligned} \mathcal{X}_M &= \{x \in \mathcal{X} \mid \operatorname{Ker}(x)/\operatorname{Im}(x) \neq 0\} \\ &= \{x \in \mathcal{X} \mid M \text{ is not projective as a } U(\langle x \rangle)\text{-module}\} \cup \{0\}. \end{aligned}$$

Here $x$ is considered as an operator from $M \to M$ by $x(m) = x.m$ with $x^2 = 0$.

# Varieties arising from the relative cohomology of $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$

Let $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, M)$ be the relative Lie algebra cohomology of the pair $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ which is obtained from the complex

$$C^\bullet = \mathrm{Hom}_{\mathfrak{g}_{\bar{0}}}(\Lambda^\bullet_{super}(\mathfrak{g}/\mathfrak{g}_{\bar{0}}), M).$$

**Theorem (BKN1)**

*Let $(\mathfrak{g}, \mathfrak{g}_{\bar{0}})$ be as above. Then*

$$\mathrm{Ext}^\bullet_{\mathcal{F}}(\mathbb{C}, \mathbb{C}) \cong \mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{g}_{\bar{0}}, \mathbb{C}) \cong (\Lambda^\bullet_{super}(\mathfrak{g}/\mathfrak{g}_{\bar{0}})^*)^{G_{\bar{0}}} \cong S^\bullet(\mathfrak{g}_{\bar{1}}^*)^{G_{\bar{0}}}.$$

Note that the cohomology ring is finitely generated because $G_{\bar{0}}$ is reductive.

Using the finite generation of cohomology we can define the following support varieties for modules in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$.

$$\mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M) = \operatorname{Maxspec}(\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{g}_{\bar{0}},\mathbb{C})/J_{\mathfrak{g}}(M \otimes M^{*})) \subseteq \mathfrak{g}_{\bar{1}}/G_{\bar{0}}$$

where $J_{\mathfrak{g}}(M \otimes M^{*})$ is the annihilator of the cohomology ring $\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{g}_{\bar{0}},\mathbb{C})$ on $\mathrm{H}^{\bullet}(\mathfrak{g},\mathfrak{g}_{\bar{0}},M \otimes M^{*})$.

**Theorem (BKN2)**

*Let* $\mathfrak{g} = \mathfrak{gl}(m|n)$, *and* $L(\lambda)$ *be a simple module in* $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$. *Then*

$$\dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(L(\lambda)) = \operatorname{atyp}(\lambda).$$

*where* $\operatorname{atyp}(\lambda)$, *is the maximal number of linearly independent mutually orthogonal, positive isotropic roots* $\alpha \in \Delta^{+}$ *such that* $(\lambda+\rho,\alpha) = 0$.

# Complexity for Kac (dual Kac) Modules: Principal Block

**Theorem (BKN4)**

*Let $K(\lambda)$ be a Kac module (resp. $K^{-}(\lambda)$ be a dual Kac module) in the principal block $\mathcal{B}_0$ of $\mathcal{F} = \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ for $\mathfrak{g} = \mathfrak{gl}(k|k)$. Then*

$$c_{\mathcal{F}}(K(\lambda)) = c_{\mathcal{F}}(K^{-}(\lambda)) = \mathrm{atyp}(\lambda)^2 = k^2.$$

# $c_{\mathcal{F}}(K(\lambda)) = k^2$

(1) Let

$$\cdots \to P_2 \to P_1 \to P_0 \to L_0(\lambda) \to 0$$

be a minimal projective resolution of $L_0(\lambda)$ in $\mathcal{F}_{(\mathfrak{p}^+,\mathfrak{g}_0)}$. Apply the exact functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} -$ to this resolution to get a projective resolution with the same rate of growth for $K(\lambda)$. This shows that

$$c_{\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}}(K(\lambda)) \leq c_{\mathcal{F}_{(\mathfrak{p}^+,\mathfrak{g}_0)}}(L_0(\lambda)) \leq \dim \mathfrak{g}_1.$$

(2) Next observe that any projective resolution in $\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}$ of a module $M$ (such as $K(\lambda)$) will restrict to a projective resolution of $M$ in $\mathcal{F}_{(\mathfrak{p}^+,\mathfrak{g}_0)}$. Therefore,

$$c_{\mathcal{F}_{(\mathfrak{p}^+,\mathfrak{g}_0)}}(M) \leq c_{\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}}(M).$$

# $c_{\mathcal{F}}(K(\lambda)) = k^2$, con't

(3) Combining these statement and using support varieties, we have

$$\dim \mathcal{V}_{\mathfrak{g}_1}^{\text{rank}}(K(\lambda)) = c_{\mathcal{F}_{(\mathfrak{p}^+,\mathfrak{g}_0)}}(K(\lambda)) \leq c_{\mathcal{F}_{(\mathfrak{g},\mathfrak{g}_0)}}(K(\lambda)) \leq k^2.$$

(4) Let $I_k$ be the identity matrix in $\mathfrak{g}_1$. As a $U(\langle I_k \rangle)$-module, $K(\lambda)$ decomposes as

$$K(\lambda)|_{U(\langle I_k \rangle)} \cong K(0) \oplus \left(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^+)} N\right).$$

Moreover,

$$K(0)|_{U(\mathfrak{p}^+)} \cong (1 \otimes \mathbb{C}) \oplus (U(\mathfrak{g}_{-1})\mathfrak{g}_{-1} \otimes \mathbb{C}) \cong \mathbb{C} \oplus (U(\mathfrak{g}_{-1})\mathfrak{g}_{-1} \otimes \mathbb{C}).$$

Since $I_k \in \mathfrak{p}^+$, it follows that $K(\lambda)$ as a $U(\langle I_k \rangle)$-module has $\mathbb{C}$ as a direct summand, which proves that $K(\lambda)$ is not free as a $U(\langle I_k \rangle)$-module.

# Complexity for Kac (dual Kac) Modules: General Case

**Theorem (BKN4)**

*Let $K(\lambda)$ be a Kac module (resp. $K^{-}(\lambda)$ be a dual Kac module) for $\mathfrak{g}l(m|n)$ with $\text{atyp}(\lambda) = k$. Then*

(a) $c_{\mathcal{F}}(K(\lambda)) = \dim \overline{(\mathfrak{g}_1)_k} = (m+n)k - k^2$;

(b) $c_{\mathcal{F}}(K^{-}(\lambda)) = \dim \overline{(\mathfrak{g}_{-1})_k} = (m+n)k - k^2$.

# $c_{\mathcal{F}}(K(\lambda)) \geq \dim \overline{(\mathfrak{g}_1)_k}$

(1) First note that if $\mathcal{B}$ is block of $\mathcal{F}$ then all simple modules in $\mathcal{B}$ has the same atypicality. So one can talk about the atypicality of a block.

(2) Gruson and Serganova show that if $\mathcal{B}$ is a block of atypicality $k$ then there is an equivalence of categories with the principal block $\mathcal{B}_0$ of $\mathfrak{gl}(k|k)$. This equivalence involves translation functors and a restriction functor. Under this equivalence Kac modules go to Kac modules.

(3) Using the Gruson-Serganova equivalence we show that

$$I_k \in \mathcal{V}_{\mathfrak{g}_1}(K(\lambda))$$

where $I_k$ is the "standard" rank $k$ matrix in $\mathfrak{g}_1$.

(4) Therefore,

$$\dim \overline{(\mathfrak{g}_1)_k} = \dim \overline{G_{\bar{0}} \cdot I_k} \leq c_{\mathcal{F}_{(\mathfrak{p}^+, \mathfrak{g}_{\bar{0}})}}(K(\lambda)) \leq c_{\mathcal{F}_{(\mathfrak{g}, \mathfrak{g}_{\bar{0}})}}(K(\lambda)).$$

# $c_{\mathcal{F}}(K(\lambda)) \leq \dim \overline{(\mathfrak{g}_1)_k}$

(1) Let $M$ be a $\mathfrak{gl}(m|n)$-module which lies in a block of atypicality $k$. Let $P_\bullet \to M$ be a minimal projective resolution for $M$. Then there is a positive constant $C$ depending only on $m$, $n$ and $M$ such that if $P(\mu)$ appears as a direct summand of $P_d$, then

$$\dim P(\mu) \leq Cd^{(m+n-k-1)k}.$$

This involves an intricate analysis using Weyl's dimension formula.

(2) Recall that

$$c_{\mathcal{F}}(K(\lambda)) = r\left(\operatorname{Ext}^\bullet_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}\left(K(\lambda), \oplus L(\mu)^{\dim P(\mu)}\right)\right),$$

where the direct sum is over all simple modules in the block which contains $K(\lambda)$.

# $c_{\mathcal{F}}(K(\lambda)) \leq \dim \overline{(\mathfrak{g}_1)_k}$, con't

(3) For fixed $d$, and set $T = \oplus L(\mu)^{\dim P(\mu)}$

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), T) = \sum \dim P(\mu) \cdot \dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), L(\mu)).$$

If $P_\bullet \to K(\lambda)$ is a minimal projective resolution, then

$$\operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), L(\mu)) = \operatorname{Hom}_{\mathcal{F}}(P_d, L(\mu))$$

being nonzero implies $P(\mu)$ is a summand of $P_d$. By (1) $\dim P(\mu) \leq C_1 d^{(m+n-k-1)k}$ for some constant $C_1$ which depends only on $m$, $n$, and $\lambda$. Thus,

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), T) \leq C_1 d^{(m+n-k-1)k} \dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), \oplus L(\mu)).$$

# $c_{\mathcal{F}}(K(\lambda)) \leq \dim \overline{(\mathfrak{g}_1)_k}$, con't

(4) Therefore, it suffices to prove

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), \oplus L(\mu)) \leq C_2 d^{k-1},$$

where $C_2$ is a constant independent of $d$. If this is true then

$$\begin{aligned}
\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), T) &\leq C_1 d^{(m+n-k-1)k} \dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), \oplus L(\mu)) \\
&\leq C_1 d^{(m+n-k-1)k} C_2 d^{k-1} \\
&= C_1 C_2 d^{(m+n)k-k^2-1} \\
&= C_1 C_2 d^{\dim \overline{(\mathfrak{g}_1)_k}-1}.
\end{aligned}$$

# $c_{\mathcal{F}}(K(\lambda)) \leq \dim \overline{(\mathfrak{g}_1)_k}$, con't

(5) In order to find the aforementioned bound

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), \oplus L(\mu)),$$

we use the facts that

(i) $\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), L(\mu))$ is the coefficient of a Kazhdan-Lusztig polynomial (Brundan), and the coefficients of these polynomials are uniformly bounded ([BKN4]).

(ii) $\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(K(\lambda), L(\mu)) \neq 0$ implies that $\mu$ is a "partition" (after tensoring by the superdet rep.) between $|\lambda| + d$ and $|\lambda| + d + k^2$. The number of partitions with no more that $k$ parts is bounded by $C_3 i^{k-1}$ where $C_3$ is a constant only depending on $k$.

# Complexity for Simple Modules

## Theorem (BKN4)

*Let $L(\lambda)$ be a simple $\mathfrak{gl}(m|n)$-module of atypicality $k$. Then*

$$c_{\mathcal{F}}(L(\lambda)) = \dim \overline{(\mathfrak{g}_1)_k} + k = (m+n)k - k^2 + k$$

# A Reduction Theorem

## Theorem

*Let $L(\lambda)$ and $L(\mu)$ be two simple modules for $\mathfrak{gl}(m|n)$ with* $\operatorname{atyp}(\lambda) = \operatorname{atyp}(\mu)$. *Then*

$$L(\lambda)^* \otimes L(\lambda) \otimes L(\mu) \cong L(\mu) \oplus U$$

*for some $\mathfrak{gl}(m|n)$-module $U$. Furthermore, the complexity of $L(\lambda)$ equals the complexity of $L(\mu)$.*

This theorem uses the Generalized Kac-Wakimoto Conjecture for basic classical Lie superalgebras (as stated by Geer, Kujawa, and Patureau-Mirand) and the verification of this conjecture for $\mathfrak{gl}(m|n)$ by Serganova.

# $c_{\mathcal{F}}(L(\lambda)) \leq \dim \overline{(\mathfrak{g}_1)_k} + k$

(1) As in the Kac module case (by using the bound on the dimension of projectives) we can reduce to showing that

$$\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}(L(\lambda), \oplus L(\mu)) \leq Cd^{2k-1},$$

where $C$ is a positive constant.

(2) By applying the previous theorem along with the Gruson-Serganova equivalence one can reduce this to the case when $L(\lambda) = \mathbb{C}$ and we are in the principal block $\mathcal{B}_0$ for $\mathfrak{gl}(k|k)$.

# $c_{\mathcal{F}}(L(\lambda)) \leq \dim \overline{(\mathfrak{g}_1)_k} + k$, con't

(3) The category $\mathcal{F}$ is a highest weight category with an "abstract" Kazhdan-Lusztig theory (terminology of Cline-Parshall-Scott). So

$$\begin{aligned} \dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}(\mathbb{C}, L(\mu)) &= \sum_{i+j=d} \sum_{\sigma \in \mathcal{B}_0} \dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma), \mathbb{C}) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)) \\ &= \sum_{i+j=d} \sum_{\sigma \in \mathcal{B}_0} \dim \operatorname{Hom}_{\mathfrak{g}_{\bar{0}}}(L_0(\sigma), S^i(\mathfrak{g}_1^*)) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)) \end{aligned}$$

We can now invoke the fact that the Kazhdan-Lusztig polynomials are bounded by a constant and the composition factors of $S^\bullet(\mathfrak{g}_1^*)$ are multiplicity free (Schmidt), so $\dim \operatorname{Hom}_{\mathfrak{g}_{\bar{0}}}(L_0(\sigma), S^i(\mathfrak{g}_1^*))$ is bounded by number of partitions of $i$ into at most $k$-parts.

# $c_{\mathcal{F}}(L(\lambda)) \geq \dim \overline{(\mathfrak{g}_1)_k} + k$

At this time there is no known support theory for $\mathcal{F}$ so we need a replacement to find a lower bound for $c_{\mathcal{F}}(L(\lambda))$. Let $\mathcal{B}$ be a block of atypicality $k$. For each $d$, we look at a specific set of pairs of highest weights $S(d)$ in a block of atypicality $k$ in $\mathfrak{gl}(m|n)$.

## Lemma

*Let $\mathcal{B}$ be the block above and let $(\mu, \sigma) \in S(d) \subset \mathcal{B} \times \mathcal{B}$. Then, for d sufficiently large,*

$$\dim P(\mu) \geq C d^{(m+n-k-1)k},$$

*where $C$ is a positive constant which is independent of $\mu$ and $\sigma$.*

# $c_{\mathcal{F}}(L(\lambda)) \geq \dim \overline{(\mathfrak{g}_1)_k} + k$, con't

By using the properties of Kazhdan-Lusztig polynomials and Schmidt's result we have

$$\sum_{i+j=d} \sum_{(\mu,\sigma)\in S(d)} \dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma), L(\nu)) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)) \geq |S(d)|.$$

# $c_{\mathcal{F}}(L(\lambda)) \geq \dim \overline{(\mathfrak{g}_1)_k} + k$, con't

Set $T = \bigoplus_{\mu\in\mathcal{B}} L(\mu)^{\dim P(\mu)}$

$$\begin{aligned}
\dim \operatorname{Ext}^d_{(\mathfrak{g},\mathfrak{g}_0)}(L(\nu), T) &= \sum_{i+j=d} \sum_{\mu,\sigma\in\mathcal{B}} \dim P(\mu) \dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma), L(\nu)) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)) \\
&\geq \sum_{i+j=d} \sum_{(\mu,\sigma)\in S(d)} \dim P(\mu) \dim \operatorname{Ext}^i_{\mathcal{F}}(K(\sigma), L(\nu)) \dim \operatorname{Ext}^j_{\mathcal{F}}(K(\sigma), L(\mu)) \\
&\geq Cd^{(m+n-k)k-k}|S(d)| \\
&\geq Cd^{(m+n-k)k-k}Q(d).
\end{aligned}$$

where $Q(d)$, of degree $2k-1$ with positive leading coefficient, by Erhart's theorem on counting lattice points in a polytope.

# A Shadow?

**Theorem (BKN4)**

*Let $K^{\pm}(\lambda)$ be a Kac (resp. dual Kac) module for $\mathfrak{gl}(m|n)$ with* $\operatorname{atyp}(\lambda) = k$. *Then*

(a) $\mathcal{X}_{K^{\pm}(\lambda)} = \mathcal{V}_{\mathfrak{g}_{\pm 1}}(K^{\pm}(\lambda)) = \overline{(\mathfrak{g}_{\pm 1})_k}$;

(b) $c_{\mathcal{F}}(K^{\pm}(\lambda)) = \dim \mathcal{X}_{K^{\pm}(\lambda)} + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_0)}(K^{\pm}(\lambda))$.

**Theorem (BKN4)**

*Let $L(\lambda)$ be a simple $\mathfrak{gl}(m|n)$-module of atypicality $k$. Then*

$$c_{\mathcal{F}}(L(\lambda)) = \dim \overline{(\mathfrak{g}_1)_k} + k = (m+n)k - k^2 + k = \dim \mathcal{X}_{L(\lambda)} + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(L(\lambda)).$$

## Conjecture

*Let* $\mathfrak{g} = \mathfrak{gl}(m|n)$ *and* $M \in \mathcal{F}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}$. *Then*

$$c_{\mathcal{F}}(M) = \dim \mathcal{X}_M + \dim \mathcal{V}_{(\mathfrak{g},\mathfrak{g}_{\bar{0}})}(M).$$

In general does there exist a natural subvariety of $\mathfrak{g}_{\bar{1}}$ of dimension equal to $k^2 + k$ for $\mathfrak{gl}(k|k)$? [i.e., dimension 2 for $\mathfrak{gl}(1|1)$, dimension 6 for $\mathfrak{gl}(2|2)$, dimension 12 for $\mathfrak{gl}(3|3)$, etc....]

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