COMMUTATIVE ALGEBRA FOR MODULAR REPRESENTATIONS OF FINITE GROUPS

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The statements below are all true, I believe. Proving them, or finding counter-examples if you think they are wrong, *is* the exercise.

Lecture I

Let k be a field of characteristic $p > 0$ and G a finite group.

(1) When $G := \mathbb{Z}/2$ and $\operatorname{char} k = 2$ the trivial representation is not a direct summand of the regular one.
(2) The group algebra kG is self-injective, and hence that a finitely generated kG-module is projective if and only if it is injective. (This is true for all kG-modules, and not only finitely generated ones.)
(3) The group algebra kG is a local ring if and only if G is a p-group.

In (4)–(6) assume G is a p-group; even elementary abelian, for simplicity. Set

$$R := kG \quad \text{and } \mathfrak{m} := \text{the maximal ideal of } R.$$

Note that $\mathfrak{m}^i = 0$ for $i \gg 0$.

(4) The socle of any non-zero R-module is non-zero.
(5) Any R-module M is part of exact sequences

$$0 \longrightarrow M_1 \longrightarrow R^\nu \xrightarrow{\varepsilon} M \longrightarrow 0$$

$$0 \longrightarrow M \xrightarrow{\iota} R^\mu \longrightarrow M_{-1} \longrightarrow 0$$

where $\nu := \operatorname{rank}_k(M/\mathfrak{m}M)$ and $\mu := \operatorname{rank}_k(\operatorname{soc}_R M)$, where $\operatorname{soc}_R M$ is the socle of M. Thus, $\varepsilon \otimes_R k$ and $\operatorname{Hom}_R(k, \iota)$ are isomorphisms, and so

$$M_1 \subseteq \mathfrak{m}R^\nu \text{ and } \operatorname{soc}_R M = \operatorname{soc}_R(R^\mu)\,.$$

The module M_1 is the first *syzygy* of M and M_{-1} is its first *cosyzygy*. The higher syzygies and cosyzygies are defined iteratively.

(6) Let M be a finitely generated R-module. In (b), the map ι is the one above.
   (a) If k is not a direct summand of M, then $\operatorname{soc}_R M \subseteq \mathfrak{m}M$.
   (b) If R is not a direct summand of M, then $\iota(M) \subseteq \mathfrak{m}R^\mu$, so $\operatorname{soc} R \cdot M = 0$.

The next series of exercises deals with the Klein four-group, $(\mathbb{Z}/2)^2$, over a field of char 2. Thus

$$R := k[(\mathbb{Z}/2)^2] \cong k[x,y]/(x^2,y^2) \quad \text{and} \quad \mathfrak{m} := (x,y)\,.$$

The aim is to describe the indecomposable R-modules of ranks 1 and 2; compare this with Questions 3 to 7 in Dave's lectures.

(7) If M is an indecomposable R-module, and neither R or k, then $\operatorname{soc}_R M = \mathfrak{m}M$.

Date: 27th - 30th July 2012, Seattle.

(8) If M is an indecomposable R-module with $\operatorname{rank}_k M = 2$, then it is cyclic and hence isomorphic to a module of the form
$$M_{a,b} := R/(ax+by) \quad \text{where } (a,b) \in k^2 \setminus \{(0,0)\}$$
One $M_{a,b} \cong M_{a',b'}$ if and only if $(a,b) = \lambda(a',b')$ for a non-zero $\lambda \in k$. Thus, the indecomposable modules of rank two are parameterized by $\mathbb{P}^1_k$.

(9) Suppose M is an indecomposable module or rank $2n+1$, for some integer $n \geq 1$. Then its syzygy module M_1 and its cosyzygy module M_{-1} have odd rank, and at least one of them has rank strictly less than that of M. It follows that M is a syzygy or a cosyzygy of k.

Conversely, every syzygy and cosyzygy of k is indecomposable of odd rank; proving the indecomposability is a bit tricky.

Lecture II

In this section, k is a commutative ring (nothing much is lost if you wish to assume k is a field). Our convention is that a graded k-module, say V, will be a collection $\{V^i\}_{i\in\mathbb{Z}}$ of k-modules indexed by $\mathbb{Z}$. The degree of an element v in V will be denoted $|v|$. Given a DG (which is an abbreviation of 'Differential Graded') object M, we write $M^\natural$ for the underlying graded object.

(1) Let A, B be graded k-algebras, and $A \otimes_k B$ the graded k-algebra, with

$$(A \otimes_k B)^n := \bigoplus_{i+j=n} A^i \otimes_k B^j \text{ and multiplication}$$

$$(a \otimes b) \cdot (a' \otimes b') := (-1)^{|b||a'|} aa' \otimes bb'$$

When A and B are (strictly) graded-commutative, so is $A \otimes_k B$.

When A and B are DG k-algebras, so is $A \otimes_k B$. And if M and N are DG modules over A and B, respectively, then $M \otimes_k N$ is a DG $A \otimes_k B$-module.

(2) The exterior algebra, say Λ, on indeterminates $\xi_1, \dots, \xi_r$ all of odd degrees is the tensor product $\Lambda_1 \otimes_k \cdots \otimes_k \Lambda_r$, where Λ_i is the exterior algebra on ξ_i.

This is false without the "signed-multiplication" on the tensor product.

(3) Let r be an element in a commutative ring R and let K be the Koszul complex on R, viewed as a DG R-algebra. Thus, as a complex of R-modules

$$K := 0 \longrightarrow R \xrightarrow{r} R \longrightarrow 0$$

and the multiplication is the obvious one. The data of a DG K-module structure on a graded R-module M is equivalent to specifying R-linear maps

$$d\colon M \to M \quad \text{and} \quad \sigma\colon M \to M$$

of degree $+1$ and -1, respectively, with the property that $d \circ \sigma + \sigma \circ d = r$.

(4) Let A be a DG algebra and $\alpha \in A^1$ an element satisfying

$$d(\alpha) = \alpha^2 \quad \text{and} \quad \alpha \cdot a = (-1)^{|a|} a \cdot \alpha \text{ for all } a \in A$$

For any DG A-module M, the graded $A^\natural$-module $M^\natural$ with differential

$$d(m) := d^M(m) + \alpha \cdot m$$

is also a DG A-module, denoted M^α.

(5) For $r = 1$ it is easy to check that the morphism $\varepsilon\colon X \to k$, defined in the lecture, is a quasi-isomorphism. The general case can be settled by taking tensor products.

(6) Let k be a field and $R := k[z_1, \dots, z_r]/(z_1^{d_1}, \dots, z_r^{d_r})$ where $d_i \geq 2$ for each i. For example, R might be the group algebra of an elementary abelian group. Let K be the Koszul complex on $z_1, \dots, z_r$, viewed as a DG algebra. Think of K as the exterior algebra over R on indeterminates $e_1, \dots, e_r$ of degree -1, with differential determined by $d(e_i) = z_i$ and the Leibniz rule.

Claim: $H^*(K)$ is an exterior algebra on the k-vector space $H^{-1}(K)$.

This can be verified as follows: Let Λ be an exterior algebra over k on indeterminates $\xi_1, \dots, \xi_r$ of degree -1, viewed as a DG algebra with zero differential. There is then a morphism of DG R-algebras

$$\Phi\colon \Lambda \to K \quad \text{with} \quad \Phi(\xi_i) := z^{d_i - 1} e_i\,.$$

This is a quasi-isomorphism: This is easy to check directly for the case $r = 1$, and the general case follows by taking tensor products over k.

Note that this argument proves more, namely, that K is quasi-isomorphic, as a DG algebra to an exterior algebra. This holds true for any complete intersection local ring R.

(7) Let $E = (\mathbb{Z}/2)^r$ and let k be a field of characteristic 2. Mimicking the construction of the functor F from the lecture, one can get an equivalence of categories

$$\mathsf{D}^{\mathsf{f}}(kE) \xrightarrow{\simeq} \mathsf{D}^{\mathsf{f}}(k[x_1, \dots, x_r])$$

where $k[x_1, \dots, x_r]$ is a DG algebra with $|x_i| = 1$ and zero differential.

Lecture III

(1) Let A be a DG k-algebra, and $f\colon M \to N$ a morphism of DG A-modules. The cone of f (viewed as a morphism of complexes) has a natural structure of a DG module over A such that the canonical exact sequence

$$0 \longrightarrow N \longrightarrow \mathsf{cone}(f) \longrightarrow \Sigma M \longrightarrow 0$$

is compatible with the A-action.

(2) Let A be a ring; it may not be commutative. We say that a complex is *perfect* if it is isomorphic (in the derived category) to a bounded complex of finitely generated projective A-modules, that is to say, to one of the form

$$0 \longrightarrow P^s \longrightarrow \cdots \longrightarrow P^t \longrightarrow 0$$

with each P^n a finitely generated projective A-module. It is not hard to prove that a complex is perfect, then it is in $\mathsf{thick}(A)$; the converse is also true.

A more precise statement is that a complex M of A-modules is in $\mathsf{thick}^n(A)$, for some $n \geq 0$, if and only if it is isomorphic in $\mathsf{D}(A)$ to a complex P with a filtration by subcomplexes

$$\{0\} \subseteq P(0) \subseteq P(1) \subseteq \cdots \subseteq P(n) = P$$

such that $P(i)/P(i-1)$ is a graded projective A-module, with zero differential. This extends verbatim to the case where A is a DG algebra, except that one has to allow M to be a direct summand of such a P.

(3) Let $R = k[z_1, \ldots, z_r]/(z_1^p, \ldots, z_r^p)$, with k a field. A complex M of R-modules is in $\mathsf{thick}(k)$ if and only if $H^*(M)$ has finite length. The same is true over any (commutative, noetherian) local ring R, and in fact much more generally.

(4) Let $E = \mathbb{Z}/2$ and k a field of characteristic 2. By Exercise 7 in Lecture II (if you did not do that exercise, this is a good time to do so) there is then an equivalence of categories

$$\mathsf{D}^{\mathsf{f}}(kE) \xrightarrow{\simeq} \mathsf{D}^{\mathsf{f}}(k[x])$$

where $k[x]$ is a DG algebra with $|x| = 1$ and zero differential. Think about the images under this functor of the indecomposable kE-modules (there are only two), and also of the Koszul complex on z. What are the kE-modules corresponding to the DG $k[x]$-modules $k[x]/(x^n)$?

(5) Let now $E = (\mathbb{Z}/2)^2$ and k a field of characteristic 2, so $kE \cong k[z_1, z_2]/(z_1^2, z_2^2)$. There is an equivalence of categories

$$\mathsf{D}^{\mathsf{f}}(kE) \xrightarrow{\simeq} \mathsf{D}^{\mathsf{f}}(k[x_1, x_2])$$

where $k[x_1, x_2]$ is a DG algebra with $|x_i| = 1$ and zero differential.

What are the DG $k[x_1, x_2]$-modules corresponding to the syzygy modules of k over kE? It is also worth thinking about the indecomposable modules

$$M_{(a_1,a_2)} = k[z_1, z_2]/(a_1 z_1 + a_2 z_2) \quad \text{for } (a_1, a_2) \in k^2\,.$$

(6) Think about the analogue of Exercises 4 and 5 for elementary abelian p-groups with $p \geq 3$.

Lecture IV

Let k be field and set

$$R := k[z_1, \dots, z_r]/(z_1^p, \dots, z_r^p)\,.$$

Thus, R might be the group algebra of an elementary abelian p-group of rank r.

(1) Let K be the Koszul complex on the elements $z_1, \dots, z_r$. Then K is evidently built out of R, in that it is in $\mathsf{thick}(R)$; the converse is also true, so $\mathsf{thick}(K) = \mathsf{thick}(R)$. One can prove this directly for $r = 1$ and settle the general case by taking suitable tensor products.

In fact $\mathsf{thick}(M) = \mathsf{thick}(R)$ for any perfect complex M with $H^*(M) \neq 0$; this is harder to prove, and is a special case of the classification of thick subcategories of perfect complexes over commutative noetherian rings, due to Mike Hopkins, and Amnon Neeman.

Henceforth, $r = 2$ and k is algebraically closed of characteristic $p = 2$. Thus R is the group algebra of the Klein four group. For any $\underline{a} = (a_1, a_2)$, set

$$R_{\underline{a}} := k[a_1 z_1 + a_2 z_2] \subset k[z_1, z_2] = R$$

Thus, $R_{\underline{a}}$ is the k-subalgebra of R spanned by the linear form $a_1 z_1 + a_2 z_2$.

(2) The *rank variety* of an R-module M is the subset of $\mathbb{A}^2(k)$ defined by

$$\mathcal{V}^{\mathsf{r}}_R(M) := \{(a_1, a_2) \mid M{\downarrow}_{R_{\underline{a}}} \text{ is not projective}\}$$

This a closed subset, in the Zariksi topology, of $\mathbb{A}^2(k)$. Compute the rank varieties of the syzygy modules of k over R, and of the indecomposable modules

$$M_{(a_1,a_2)} = k[z_1, z_2]/(a_1 z_1 + a_2 z_2) \quad \text{for } (a_1, a_2) \in k^2\,.$$

(3) As in my lecture, one can associate another variety to M via the equivalence $\mathsf{D}^{\mathsf{f}}(R) \xrightarrow{\simeq} \mathsf{D}^{\mathsf{f}}(k[x_1, x_2])$; this is the *support variety* of M and denoted $\mathcal{V}_R(M)$.

Compute the support varieties of the syzygy modules of k over R, and of the indecomposable modules $M_{(a_1,a_2)}$ from (2).

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