COMMUTATIVE ALGEBRA FOR MODULAR REPRESENTATIONS OF FINITE GROUPS

The following notes have been taken from a lecture series by Srikanth B. Iyengar given during a summer school on *Cohomology and Support in Representation Theory* which took place in Seattle in 2012.

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Lecture 1

The aim of this lecture series is to build up the following connections

$$\begin{tikzcd}
\text{Representations of finite groups} \arrow[rr, dashed, leftrightarrow] \arrow[dr, dashed] & & \text{Modules over the polynomial ring} \arrow[ddl, dashed, leftrightarrow, bend left] \\
& \text{Representations of elementary abelian groups} \arrow[ur, dashed] \arrow[d, phantom, "\cap"] & \\
& \text{Modules over complete intersections} &
\end{tikzcd}$$

1.1. **Group representations.** Let G be a finite group and let k be a field of characteristic $p \geq 0$. A *(k-linear) representation* of G is a k-vector space V with a G-action. This is the same as specifying a group homomorphism $G \to \mathrm{GL}_k(V)$. Easy examples are given by the zero-representation (i.e. $V = 0$) and the *trivial* representation of G, that is k with trivial G-action.

If V and W are representations of G, then so is their direct sum $V \oplus W$, namely via the G-action given by $g(v, w) := (gv, gw)$ $(g \in G,\ v \in V,\ w \in W)$. A representation $V \neq 0$ of G is *indecomposable* if $V = V_1 \oplus V_2$ for two representations V_1, V_2 of G, implies that $V_1 = 0$ or $V_2 = 0$.

Fix a *finite dimensional* representation V of G (i.e. $\dim_k(V) < \infty$). One can decompose V as

$$V = \bigoplus_{i=1}^{n} W_i^{e_i},$$

for some integers $e_i \geq 1$ and indecomposable representations W_i of G with $W_i \ncong W_j$ for $i \neq j$ $(1 \leq i, j \leq n)$. A theorem of Krull-Remak-Schmidt tells us, that such a decomposition is unique, i.e. the W_i and e_i are determined by the given representation V.

Theorem 1.1 (Maschke)**.** *If* $\mathrm{char}(k)$ *does not divide* $|G|$*, then every indecomposable representation of* G *is a direct summand of the regular representation, that is, it is* *a direct summand of the G-representation V_G given by the data:*

$$V_G := \bigoplus_{g\in G} kg, \qquad h(\sum_{g\in G} \lambda_g g) = \sum_{g\in G} \lambda_g hg, \quad h \in G.$$

Corollary 1.2. *If $\mathrm{char}(k)$ does not divide $|G|$, then there are only finitely many non-isomorphic indecomposable representations of G.*

Example 1.3. Consider the *Klein four-group* $G = \mathbb{Z}/2 \times \mathbb{Z}/2$. Let $\mathrm{char}(k) = 2$. Then the trivial representation is not a direct summand of the regular one. This follows from $\mathrm{Ext}^1_G(k,k) \neq 0$. Moreover, for any even $n \geq 2$ there are infinitely many non-isomorphic indecomposable representations of G having dimension n.

1.2. **The group algebra of G.** The regular representation V_G of G is in fact a k-algebra.

Definition 1.4. The *group algebra* kG of G is the k-vector space

$$kG := \bigoplus_{g\in G} kg \ (= V_G)$$

with multiplication induced by the product on G:

$$\mu : kG \otimes_k kG \to kG, \ \sum_{g,h\in G} \lambda_{g,h}(g\otimes h) \mapsto \sum_{g,h\in G} \lambda_{g,h} gh.$$

Note that the unit of kG is the unit of G and that kG is commutative if and only if G is abelian.

Example 1.5. Let $G = \mathbb{Z}/d = \langle g \mid g^d = 1\rangle$, then

$$kG = \frac{k[g]}{(g^d - 1)} \ .$$

More generally,

$$k[\mathbb{Z}/d_1^{e_1} \times \cdots \times \mathbb{Z}/d_r^{e_r}] = \frac{k[g_1, \ldots, g_r]}{(g_1^{e_1} - 1, \ldots, g_r^{e_r} - 1)} \ .$$

Remark 1.6. One should note that

- specifying a group homomorphism $G \to \mathrm{GL}_k(V) \cong \mathrm{Aut}_k(V)$ is the same as specifying a k-algebra homomorphism $kG \to \mathrm{End}_k(V)$. This translates to the statement, that, for a k-vector space V, having a G-action on V is the same as having a (left) kG-module structure on V.
- the map $\varepsilon : kG \to k$, $\varepsilon(g) = 1$, is a k-algebra homomorphism.

1.3. **Reduction to elementary abelian p-groups.** A p-subgroup E of G is called an *elementary abelian p-subgroup* if it is isomorphic to a group of the form

$$\mathbb{Z}/p \times \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p = (\mathbb{Z}/p)^r$$

for some $r \geq 0$. The number r is the *rank* of the elementary abelian p-subgroup. It is known, that many properties of a given kG-module can be checked by looking at its kE-module structure for every elementary abelian p-subgroup $E \subseteq G$.

Fix a subgroup H of G. The group algebra kH is then a (unital) subalgebra of kG. Let M be a kG-module. Then

$$M\downarrow_H \ := \ M \text{ as a } kH\text{-module via } kH \hookrightarrow kG.$$

The motivating theorem is the following.

Theorem 1.7. *A kG-module M is projective if and only if $M\downarrow_E$ is a projective kE-module for every elementary abelian p-subgroup $E \subseteq G$.*

Let $E = (\mathbb{Z}/p)^r = \langle g_1, \dots, g_r \mid g_i^r \rangle$ and $\operatorname{char}(k) = p > 0$. Then

$$kE = \frac{k[g_1, \dots, g_r]}{(g_1^p - 1, \dots, g_r^p - 1)} = \frac{k[z_1, \dots, z_r]}{(z_1^p, \dots, z_r^p)},$$

where $z_i = g_i - 1$. For this, note that $(a+b)^p = a^p + b^p$. Suppose $p = 2$. Then

$$kE = \frac{k[z_1, \dots, z_r]}{(z_1^2, \dots, z_r^2)},$$

which is a Koszul algebra. By definition, its *Koszul dual* is given by $\operatorname{Ext}^*_{kE}(k,k) = k[x_1, \dots, x_r]$, $|x_i| = 1$. J. Moore and S. Priddy showed, that there is an equivalence of categories:

$$\mathcal{D}^f(kE) \to \mathcal{D}^f(k[\underline{x}])$$

sending k to $k[\underline{x}]$ and kE to k. Here

$\mathcal{D}^f(kE) := \{X \in \mathcal{D}(kE) \mid H^*(X)$ finitely generated as a kE-module$\}$,

$\mathcal{D}^f(k[\underline{x}]) :=$ derived cat. of differential graded $k[\underline{x}]$-modules with f.g. cohomology,

where $k[\underline{x}]$ is viewed as a DG algebra with $\partial^{k[\underline{x}]} = 0$. Suppose now that $\operatorname{char}(k) \geq 3$. Then kE is no longer Koszul. Its Koszul dual is given by

$$\operatorname{Ext}^*_{kE}(k,k) = (\Lambda_k \bigoplus_{i=1}^r kx_i) \otimes_k k[y_1, \dots, y_r], \quad |y_i| = 2.$$

There is functor

$$F : \mathcal{D}^f(kE) \to \mathcal{D}^f(k[y_1, \dots, y_r])$$

mapping kE to k and k to $\operatorname{Ext}^*_{kE}(k,k)$. We are going to construct F in the following lectures.

Lecture 2

As before, let k be a field with $\operatorname{char}(k) = p \geq 0$ and $E := (\mathbb{Z}/p)^r$ for some $r \geq 1$.

2.1. **DG modules over DG algebras.** Let R be a commutative ring and let $M = (M, \partial^M)$ be a complex of R-modules:

$$\cdots \xrightarrow{\partial^M} M^{i-1} \xrightarrow{\partial^M} M^i \xrightarrow{\partial^M} M^{i+1} \xrightarrow{\partial^M} \cdots.$$

Denote by $M^\natural$ the underlying graded R-module $\{M^i\}_{i\in\mathbb{Z}}$. For $m \in M^i$ let $|m| := i$ be its *degree*. By a *DG (Differential Graded) R-algebra A*, we mean

(1) A is a complex of R-modules.
(2) $A^\natural$ is a graded R-algebra.
(3) The above structures satisfy the Leibniz rule:

$$\partial^A(ab) = \partial^A(a)b + (-1)^{|a|} a \partial^A(b),$$

where $a, b \in A$ are homogeneous.

Let A be a DG R-algebra. A *DG A-module M* is given as follows.

(1) M is a complex of R-modules.
(2) $M^\natural$ is a graded $A^\natural$-module.
(3) The above structures satisfy the Leibniz rule:
$$\partial^M(am) = \partial^A(a)m + (-1)^{|a|}a\partial^M(m),$$
where $a \in A,\ m \in M$ are homogeneous.

Example 2.1. (1) A graded R-algebra A can be viewed as a DG algebra with $\partial^A = 0$. Then a DG A-module is a graded A-module $\{M^i\}_{i\in\mathbb{Z}}$ along with R-linear maps $\partial^M : M^i \to M^{i+1}$, $i \in \mathbb{Z}$, such that $\partial^M \circ \partial^M = 0$ and $\partial^M(am) = (-1)^{|a|}a\partial^M(m)$ ($a \in A,\ m \in M$ homogeneous).
If $A = A^0$, then a DG A-module is simply a complex of A-modules.

(2) Fix $r \in R$. Consider the *Koszul DG algebra* $K(r)$ *on* r:
$$K(r) := \qquad \underset{}{0} \longrightarrow \underset{-1}{R} \xrightarrow{r\cdot} \underset{0}{R} \longrightarrow 0\ .$$
This has a canonical structure of a DG R-algebra. An alternative construction is given in terms of the exterior algebra:
$$K(r) := \Lambda_R(Re), \quad |e| = -1,\ \partial^{K(r)}(e) = r.$$
It is an easy exercise to show that if A and B are DG R-algebras, then the complex $A \otimes_R B$ is a DG R-algebra via
$$(a \otimes b)(a' \otimes b') := (-1)^{|b||a'|}aa' \otimes bb'.$$
The maps $A \to A \otimes_R B,\ a \mapsto a \otimes 1$ and $B \to A \otimes_R B,\ b \mapsto 1 \otimes b$ are morphisms of DG R-algebras.
Now if $\underline{r} := (r_1, \dots, r_n)$ is a sequence of elements in R, set
$$\begin{aligned} K(\underline{r}) &= K(r_1) \otimes_R \cdots \otimes_R K(r_n) \\ &= \Lambda_R\Big(\bigoplus_{i=1}^n Re_i\Big), \quad |e_i| = -1,\ \partial^{K(\underline{r})}(e_i) = r_i. \end{aligned}$$
$K(\underline{r})$ enfolds as
$$K(\underline{r}) := \qquad 0 \longrightarrow \underset{-n}{Re_1 \wedge \cdots \wedge e_n} \longrightarrow \dots \longrightarrow \underset{-1}{\textstyle\bigoplus_{i,j} Re_i \wedge e_j} \longrightarrow \underset{0}{\textstyle\bigoplus_i Re_i} \longrightarrow 0\ .$$

2.2. A DG R-algebra is *graded commutative* if $ab = (-1)^{|a||b|}ba$ for all homogeneous $a, b \in A$. Note that if the degree of $a \in A$ is odd, then $2a^2 = 0$. The graded commutative DG R-algebra A is *strictly graded commutative* if $a^2 = 0$ for all homogeneous $a \in A$ of odd degree. Note that if $A = A^{\text{even}}$, then graded commutativity is the same as strict graded commutativity. Moreover, if A and B are (strictly) graded commutative it follows that $A \otimes_R B$ is (strictly) graded commutative.

Example 2.3. $K(r_1, \dots, r_n)$ is strictly graded commutative.

Let A be a DG R-algebra. A *twisting cochain* in A is an element $\alpha \in A^1$ such that

(1) $\partial^A(\alpha) = \alpha^2$,

(2) $\alpha a = (-1)^{|a|} a\alpha$ for all homogeneous $a \in A$.

Let α be a twisting cochain. Let M be a DG A-module. The complex

$$M^\alpha = (M^\natural, \partial^M + \alpha\cdot)$$

delivers a DG A-module M^α. In cash, its differential is given by

$$\partial^{M^\alpha}(m) = \partial^M(m) + \alpha m, \quad m \in M.$$

2.2. **The derived category of DG modules.** Let A be a DG R-algebra and M a DG A-module. Then $H^*(A)$ is a graded R-algebra and $H^*(M)$ is a graded $H^*(A)$-module.

A morphism $f : M \to N$ of DG A-modules (i.e. a morphism of graded A-modules which commutes with the differentials) is a *quasi-isomorphism* if $H^*(f) : H^*(M) \to H^*(N)$ is an isomorphism. Then

$$\mathcal{D}(A) := (\text{DG } A\text{-modules})[\text{quasi-iso}^{-1}]$$

is the *derived category of DG A-modules*. Its *suspension* is given as follows.

If M is a DG A-module, denote by ΣM the DG A-module whose underlying graded A-module is given by

$$\Sigma M^i = M^{i+1}, \quad i \in \mathbb{Z},$$

with A acting via

$$a \star m := (-1)^{|a|} am, \quad a \in A,\ m \in M \text{ homogeneous}.$$

The differential is $\partial^{\Sigma M} = -\partial^M$. ΣM is the *suspension* of M. We obtain a functor $\Sigma(?)$ being an equivalence of categories and delivering an automorphism of $\mathcal{D}(A)$ which we are also going to denote by Σ. Define $\Sigma^{i+1}M = \Sigma(\Sigma^i M)$, $i \in \mathbb{Z}$.

We go back to our leading example, namely the group algebra of $E = (\mathbb{Z}/p)^r$:

$$kE = \frac{k[z_1, \dots, z_r]}{(z_1^p, \dots, z_r^p)} .$$

Let K be the Koszul DG algebra on $z_1, \dots, z_r$, i.e.

$$K = \Lambda_{kE}(\bigoplus_{i=1}^r kEe_i), \quad |e_i| = -1,\ \partial^K(e_i) = z_i.$$

Evidently: $\partial^K(z_i^{p-1}e_i) = z_i^{p-1}z_i = z^p = 0$. It is a *fact*, that

$$H^{-1}(K) = \bigoplus_{i=1}^r k[z_i^{p-1}e_i]$$

and

$$H^*(K) = \Lambda_k(\Sigma(H^{-1}(K))).$$

This is the crucial property of the Koszul DG algebra of kE.

2.4. Set $S := k[y_1, \dots, y_r]$, $|y_i| = 2$ for $i = 1, \dots, r$. We view S as a DG algebra with $\partial^S = 0$. Set $A := K \otimes_k S$ which is a DG k-algebra being strictly graded commutative. Put

$$\alpha := \sum_{i=1}^r z_i^{p-1} e_i \otimes y_i \ \in \ A^1.$$

One observes that
$$\partial^A(\alpha) = \sum_{i=1}^{r} \partial^K(z_i^{p-1}e_i) \otimes y_i = 0 = \alpha^2.$$
Therefore α is a twisting cochain. Denote by S^* the DG S-module
$$(S^*)^\natural = \operatorname{Hom}_k(S,k), \quad \partial^{S^*} = 0.$$
Then $K \otimes_k S^*$ is a DG A-module. Set
$$X := (K \otimes_k S^*)^\alpha.$$

Example 2.5. Consider the case $r = 1$. Then $S = k[y]$ with $|y| = 2$. We have that $S = k[y^{-1}]$ and the S-module structure is given by $y.y^{-j} = y^{-j+1}$ if $j \geq 1$, $y.1 = 0$.

Remember that the Koszul DG algebra of $kE = k(z)/(z^p)$ on z looks as follows:
$$\begin{array}{ccccccc} 0 \longrightarrow & kE & \xrightarrow{z\cdot} & kE & \longrightarrow 0 \, . \\ & -1 & & 0 & \end{array}$$
One may think of X as
$$\begin{array}{ccccccccc} \cdots \longrightarrow & kEey^{-1} & \longrightarrow & kEy^{-1} & \longrightarrow & kEe & \longrightarrow & kE & \longrightarrow 0 \, . \\ & -3 & & -2 & & -1 & & 0 & \end{array}$$

2.6. There is a natural map $\varepsilon : X \to k$ that is k-linear. Note the following two useful *facts*.

1. (1) ε is a quasi-isomorphism. Hence X is exact in every degree different from zero.
2. (2) X^i is a free kE-module for all $i \leq 0$ and $X^i = 0$ for $i > 0$, that is X is a free resolution of k over kE with augmentation given by ε.

The assignment
$$\operatorname{Mod}(kE) \ni \ M \mapsto \operatorname{Hom}_{kE}(X, M) \ \in \operatorname{Mod}(S)$$
induces an exact functor
$$\mathcal{D}(kE) \to \mathcal{D}(S).$$

Theorem 2.7 (Avramov, Buchweitz, I., Miller)**.** *The above functor induces an exact functor*
$$F : \mathcal{D}^f(kE) \to \mathcal{D}^f(S)$$
such that
$$F(kE) = k \quad \text{and} \quad F(k) = \bigoplus_{i=0}^{r} \Sigma^{-i} S^{\binom{r}{i}}.$$

Here we used the following notation:
$$\mathcal{D}^f(kE) := \{M \in \mathcal{D}(kE) \mid H^*(M) \text{ finitely generated as a } kE\text{-module}\},$$
$$\mathcal{D}^f(S) := \{N \in \mathcal{D}(S) \mid H^*(S) \text{ finitely generated as a } S\text{-module}\}.$$

Remark 2.8. (1) $H^*(F(M)) \cong \operatorname{Ext}^*_{kE}(k, M)$ and $S \subseteq \operatorname{Ext}^*_{kE}(k,k) \cong F(k)$ as S-modules.

(2) F is faithful on objects, but not on maps.
(3) F gives rise to the following commutative diagram:

$$\begin{tikzcd}
\mathrm{thick}_{kE}(kE) \arrow[r, "F"] \arrow[d, "\subseteq"'] & \mathrm{thick}_S(k) \arrow[d, "\subseteq"] \\
\mathcal{D}^f(kE) \arrow[r, "F"] \arrow[d, two heads] & \mathcal{D}^f(S) \arrow[d, two heads] \\
\mathrm{stmod}(kE) \arrow[r, "\text{'}F\text{'}"] & \mathrm{Diff}(\mathbb{P}^{r-1})
\end{tikzcd}$$

where $\mathrm{Diff}(\mathbb{P}^{r-1})$ is the category of differential sheaves on $\mathbb{P}^{r-1}$. The categories $\mathrm{thick}_{kE}(kE)$ and $\mathrm{thick}_S(k)$ will be constructed in lecture 3.

Lecture 3

Let k be a field with $\mathrm{char}(k) = p \geq 0$, let $R = kE$, $E = (\mathbb{Z}/p)^r$, as before and put

$$S := k[y_1, \ldots, y_r], \quad |y_i| = 2, \quad \partial^S = 0.$$

3.1. Let M be a R-module. The *Loewy length of M* is

$$\ell\ell_R(M) := \inf\{n \geq 0 \mid (\underline{z})^n M = 0\}.$$

Note that if $\ell := \ell\ell_R(M)$, we obtain a filtration

$$0 = (\underline{z})^\ell M \subsetneq (\underline{z})^{\ell-1} M \subsetneq \cdots \subsetneq (\underline{z}) M \subsetneq M$$

where each subquotient

$$\frac{(\underline{z})^i M}{(\underline{z})^{i+1} M}, \quad 0 \leq i \leq \ell - 1,$$

is a k-vector space. Note that $\ell\ell_R(M) \leq \ell\ell_R(R) < \infty$.

Theorem 3.2. *For a given a perfect complex $P^\bullet$ over R, that is a complex*

$$0 \to P^s \to \cdots \to P^t \to 0$$

with P^i a finitely generated projective R-module ($s \leq i \leq t$), with $H^*(P^\bullet) \neq 0$, one has that

$$\sum_{i \in \mathbb{Z}} \ell\ell_R(H^i(P^\bullet)) \geq r + 1.$$

We are going to prove this theorem. Beforehand, we will discuss some applications and introduce further notation.

3.3. Suppose that E acts freely on a topological space X. Then the associated complex $c_*(X, k)$ is a perfect one. Therefore, by the theorem,

$$\sum_{i \in \mathbb{Z}} \ell\ell_{kE}(H_i(X, k)) \geq r + 1.$$

In particular, if the E-action on $H_i(X, k)$ is trivial, then

$$\#\{i \mid H_i(X, k) \neq 0\} \geq r + 1.$$

From this we deduce that $(\mathbb{Z}/2)^r$ cannot act freely on S^n for $r \geq 2$.

3.4. For the moment, put
$$R := \frac{k[\![z_1,\dots,z_c]\!]}{(f_1,\dots,f_c)},$$
where $f_1,\dots,f_c$ is a regular sequence (for example: $f_i = z_i^{d_i}$ for some $d_i \geq 1$). Theorem 3.2 extends to such rings. In particular, $\ell\ell_R(R) \geq c+1$ holds, so $(\underline{z})^c \neq 0$ in R, i.e. $(\underline{z})^c \nsubseteq (f_1,\dots,f_c)$ in $k[\![z_1,\dots,z_c]\!]$.

Compare this to the *New Intersection Theorem*: If Λ is a local ring and $P^\bullet$ a perfect complex over Λ with $0 < \mathrm{length}(H^*(P^\bullet)) < \infty$, then $(t-s) \geq \mathrm{Kdim}(\Lambda)$. Here $s \leq t$ are integers such that $P^i = 0$ for $i \notin \{s, s+1, \dots, t\}$.

3.5. Let us recall some further constructions on DG algebras and DG modules. Let A be a DG algebra.

(1) Remember, that the direct sum of two DG A-modules becomes a DG A-module in the obvious way.
(2) The mapping cone $\mathrm{Cone}(f)$ of a DG A-module homomorphism $f : M \to N$ is a DG A-module such that the natural sequence
$$0 \to N \to \mathrm{Cone}(f) \to \Sigma M \to 0$$
is an exact sequence of DG A-modules.

Mapping cones define the exact triangles in $\mathcal{D}(A)$, i.e.
$$\Delta : L \xrightarrow{f} M \to N \to \Sigma L,$$
where N is the mapping cone of f (up to an isomorphism in $\mathcal{D}(A)$).

3.1. **Thickenings.** Let A be a DG algebra und C a DG A-module. Consider the following sequence
$$\mathrm{thick}_A^0(C) \subseteq \mathrm{thick}_A^1(C) \subseteq \mathrm{thick}_A^2(C) \subseteq \cdots \subseteq \bigcup_{n\geq 1} \mathrm{thick}_A^n(C) =: \mathrm{thick}_A(C)$$
of full subcategories of $\mathcal{D}(A)$.

- $\mathrm{thick}_A^0(C) = \{0\}$.
- $M \in \mathcal{D}(A)$ lies in $\mathrm{thick}_A^1(C)$ if and only if M is a direct summand of
$$\bigoplus_{i=1}^{t} \Sigma^i C^{b_i}$$
for some integers $t, b_1, \dots, b_t \geq 1$.
- Let $n \geq 2$. $M \in \mathcal{D}(A)$ lies in $\mathrm{thick}_A^n(C)$ if and only if there is an exact triangle
$$\Delta : N \xrightarrow{f} L \to M \oplus M' \to \Sigma N$$
in $\mathcal{D}(A)$ such that $N \in \mathrm{thick}_A^1(C)$ and $L \in \mathrm{thick}_A^{n-1}(C)$.

The DG A-modules in $\mathrm{thick}_A(C)$ are those, that *can be build out of* C. Let Λ be any ring, i.e. a DG algebra concentrated in degree zero. Then $\mathrm{thick}_\Lambda(\Lambda)$ computes as follows.

- $\mathrm{thick}_\Lambda^1(\Lambda) =$ finitely generated graded projective Λ-modules, i.e. complexes of finitely generated Λ-modules with zero differential.
- $\mathrm{thick}_\Lambda^2(\Lambda) = \{\mathrm{Cone}(P \to Q) \mid P, Q \in \mathrm{thick}_\Lambda^1(\Lambda)\}$.

- In general: $\mathrm{thick}^1_\Lambda(\Lambda)$ = perfect complexes of Λ-modules. In fact, $M \in \mathrm{thick}^n_\Lambda(\Lambda)$ if and only if M is isomorphic (in $\mathcal{D}(\Lambda)$) to a complex $P^\bullet$ that admits a filtration by subcomplexes
$$0 = P^\bullet(0) \subseteq P^\bullet(1) \subseteq \cdots P^\bullet(n-1) \subseteq P^\bullet(n) = P^\bullet$$
such that the subquotients
$$\frac{P^\bullet(i+1)}{P^\bullet(i)}, \quad 0 \le i \le n-1,$$
are finitely generated graded projectives.

Definition 3.6. Let A be a DG algebra. A DG A-module M is *perfect* if $M \in \mathrm{thick}_A(A)$.

Remark 3.7. If A is noetherian, then $\mathrm{thick}_A(A) \subseteq \mathcal{D}(A)$. Equality holds if and only if A is regular, i.e. $\mathrm{gldim}(A) < \infty$.

3.2. **Levels.** Let A be a DG algebra and $C, M \in \mathcal{D}(A)$. In case $M \in \mathrm{thick}_A(C)$, set
$$\mathrm{level}^C_A(M) := \inf\{n \ge 0 \mid M \in \mathrm{thick}^n_A(C)\}.$$
If $M \notin \mathrm{thick}_A(C)$, put $\mathrm{level}^C_A(M) := \infty$. Observe that, if $F : \mathcal{D}(A) \to \mathcal{D}(B)$ is an exact functor, where B is a DG algebra, then
$$\mathrm{level}^C_A(M) \ge \mathrm{level}^{F(C)}_B(F(M)).$$

Remark 3.8. If $H^*(A)$ is noetherian and $H^*(M)$ is a finitely generated $H^*(A)$-module, then $1 + \mathrm{gldim}(H^*(A)) \ge \mathrm{level}^A_A(M)$. In particular,
$$r + 1 \ge \mathrm{level}^S_S(M)$$
for any $M \in \mathcal{D}^f(S)$. Hence $\mathrm{thick}(S) = \mathcal{D}^f(S)$.

3.9. Now let R be kE again (or any local ring with residue field k). Let M be a finitely generated R-module and $\ell = \ell\ell_R(M)$. Recall that there is a filtration
$$0 = (\underline{z})^\ell M \subsetneq (\underline{z})^{\ell-1} M \subsetneq \cdots \subsetneq (\underline{z}) M \subsetneq M$$
where each subquotient
$$\frac{(\underline{z})^i M}{(\underline{z})^{i+1} M}, \quad 0 \le i \le \ell - 1,$$
is a k-vector space, i.e. in $\mathrm{thick}^1_R(k)$. Hence $\mathrm{level}^k_R(M) \le \ell\ell_R(M)$ (in fact, equality holds). More generally,
$$\mathrm{level}^k_R(M) \le \sum_{i \in \mathbb{Z}} \ell\ell_R(H^i(M)), \quad M \in \mathcal{D}^f(R).$$
We have established all necessary results and notation to prove Theorem 3.2.

Proof of Theorem 3.2. If $P^\bullet$ is a perfect complex, then $P^\bullet \in \mathrm{thick}_R(R)$. Therefore $F(P^\bullet) \in \mathrm{thick}_S(F(R)) = \mathrm{thick}_S(k)$. Combining this with the faithfulness of F, we get
$$0 < \mathrm{length}_S H^*(F(P^\bullet)) < \infty$$

Moreover,

$$\begin{aligned}\sum_{i\in\mathbb{Z}} \ell\ell_R H^i(P^\bullet) &\geq \operatorname{level}_R^k(P^\bullet)\\ &\geq \operatorname{level}_S^{F(k)}(F(P^\bullet))\\ &= \operatorname{level}_S^S(F(P^\bullet)) \qquad (\text{for } \operatorname{thick}_S^1(F(k)) = \operatorname{thick}_S^1(S))\\ &\geq \operatorname{Kdim}(S)+1,\end{aligned}$$

where the last inequality uses the DG algebra version of the New Intersection Theorem stated after the proof. □

Theorem 3.10. *If S is a DG algebra such that $\partial^S = 0$ and S is commutative and noetherian containing a field, then for any DG S-module M one has*

$$\operatorname{level}_S^S(M) \geq \operatorname{Kdim}(S)+1$$

Lecture 4

Let k be a field with $\operatorname{char}(k) = p \geq 0$. As usual, let

$$R := \frac{k[z_1,\dots,z_r]}{(z_1^p,\dots,z_r^p)},$$

$$S := k[y_1,\dots,y_r], \quad |y_i| = 2, \quad \partial^S = 0.$$

Let $K(\underline{z})$ be the Koszul DG algebra on $\underline{z} = (z_1,\dots,z_r)$. For any R-module or complex M over R put $K(\underline{z};M) := K(\underline{z})\otimes_R M$. Using this, F may be expressed as

$$F(M) = (S\otimes_k K(\underline{z};M))^\alpha,\ \alpha = \sum_{i=1}^r y_i\otimes z_i^{p-1}e_i \quad (M\in\mathcal{D}^f(R)).$$

It is a *fact* that F admits a left adjoint G:

$$\begin{tikzcd} \mathcal{D}^f(R) \arrow[d, shift left, "F"] \\ \mathcal{D}^f(S) \arrow[u, shift left, "X\otimes_S^{\mathbb{L}}?=:G"] \end{tikzcd}$$

Theorem 4.1. (1) *$GF(M) = K(\underline{z};M)$ for all $M\in\mathcal{D}^f(R)$.*
(2) *$FG(N) = \bigoplus_{i=0}^r \Sigma^{-i}N^{\binom{n}{r}}$ for all $N\in\mathcal{D}^f(S)$.*

Corollary 4.2. (1) *$\operatorname{thick}_R(M) = \operatorname{thick}_R(GF(M))$ for all $M\in\mathcal{D}^f(R)$.*
(2) *$\operatorname{thick}_S(N) = \operatorname{thick}_S(FG(N))$ for all $N\in\mathcal{D}^f(S)$.*

Proof. The proof of (2) is clear. Main fact used for (1):

$$\operatorname{thick}_R(R) = \operatorname{thick}_R(K(\underline{z})),$$

where the inclusion $\supseteq$ is easy ($K(\underline{z})$ is perfect over R), while $\subseteq$ is not. It follows, that

$$\operatorname{thick}_R(M) = \operatorname{thick}_R(K(\underline{z})\otimes_R M).$$

□

4.3. The corollary delivers the bijection in the top row of the following diagram:

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Thick subcategories}\\ \text{of } \mathcal{D}^f(R)\end{matrix}\right\} \arrow[r, leftrightarrow, "\sim"] \arrow[dr, dashed, leftrightarrow, "\sim"'] & \left\{\begin{matrix}\text{Thick subcategories}\\ \text{of } \mathcal{D}^f(S)\end{matrix}\right\} \arrow[d, leftrightarrow, "\sim"] \\
 & \left\{\begin{matrix}\text{Specialization closed}\\ \text{subsets of } \operatorname{Spec}(S)\end{matrix}\right\}
\end{tikzcd}$$

where the vertical bijection is due to Hopkins and Benson-I.-Krause. This recovers a result by Benson-Carlson-Rickhard represented by the dashed arrow.

4.1. **Supports.** From now on, assume that $k = \overline{k}$. By Hilbert's Nullstellensatz, we know that the maximal ideals of S are in one-to-one correspondence to $\mathbb{A}^r$. Let $I \subseteq S$ be a homogeneous ideal. Then $V(I)$ is a cone in $\mathbb{A}^r$. For $\underline{a} \in \mathbb{A}^r$, let $\ell\underline{a}$ be the line through $\underline{0}$ and $\underline{a}$ inside $\mathbb{A}^r$. We have that

$$\underline{a} \in \mathbb{A}^r \text{ lies in } V(I) \iff \ell\underline{a} \cap V(I) \neq \{\underline{0}\}.$$

This may be expressed algebraically. For $\underline{a} \in \mathbb{A}^r$, consider

$$\pi_{\underline{a}} : S = k[y_1, \dots, y_r] \to k[y], \ y_i \mapsto a_i y.$$

Then, $\underline{a} \in V(I)$ if and only if $\dim_k(k[y]_{\underline{a}} \otimes_S S/I) = \infty$. Here $k[y]_{\underline{a}}$ denotes $k[y]$ with S-module structure coming from $\pi_{\underline{a}}$.

Definition 4.4. Let $M \in \mathcal{D}(R)$. Set

$$V_R(M) := \{\underline{a} \in \mathbb{A}^r \mid \dim_k H^*(k[y] \otimes_S F(M)) = \infty\}.$$

Theorem 4.5 (Avramov)**.** *Fix $\underline{a} \in \mathbb{A}^r \setminus \{\underline{0}\}$. Consider the canonical map*

$$R\underline{a} := \frac{k[z_1, \dots, z_r]}{(a_1 z_1^p + \cdots + a_r z_r^p)} \longrightarrow \frac{k[z_1, \dots, z_r]}{(z_1^p, \dots, z_r^p)} = R.$$

Then for any $M \in \mathcal{D}^f(R)$ one has:

$$\underline{a} \in V_R(M) \iff M\downarrow_{R\underline{a}} \notin \operatorname{thick}_{R\underline{a}}(R\underline{a}) \quad (\textit{i.e. } \operatorname{pd}_{R\underline{a}}(M\downarrow_{R\underline{a}}) = \infty)$$

Proof.

$$\begin{aligned}
\underline{a} \in V_R(M) &\iff \dim_k H^*(k[y]_{\underline{a}} \otimes_S (S \otimes_k K(\underline{z}; M))^{\alpha}) = \infty \\
&\iff \dim_k H^*(k[y] \otimes_k K(\underline{z}; M)^{\alpha_{\underline{a}}}) = \infty,
\end{aligned}$$

where

$$\alpha_{\underline{a}} := \sum_{i=1}^r a_i y \otimes z_i^{p-1} e_i = y \otimes \sum_{i=1}^r a_i z_i^{p-1} e_i \in (k[y] \otimes K(\underline{z}; M))^1.$$

There is a functor $f_{\underline{a}} : \mathcal{D}^f(R\underline{a}) \to \mathcal{D}^f(k[y])$ defined similarly to F. It fits into the diagram

$$\begin{tikzcd} \mathcal{D}^f(R) \arrow[r, "F"] \arrow[d] & \mathcal{D}^f(S) \arrow[d, "k[y]_{\underline{a}} \otimes^{\mathbb{L}}_S ?"] \\ \mathcal{D}^f(R\underline{a}) \arrow[r, "f_{\underline{a}}"'] & \mathcal{D}^f(k[y]) \end{tikzcd}$$

and fulfills $(k[y] \otimes_k K(\underline{z}; M))^{\alpha_{\underline{a}}} = f_{\underline{a}}(M \downarrow_{R\underline{a}})$. We conclude:

$$\begin{aligned} \underline{a} \in V_R(M) &\iff f_{\underline{a}}(M \downarrow_{R\underline{a}}) \notin \mathrm{thick}_{k[y]}(k) \\ &\iff M \downarrow_{R\underline{a}} \notin \mathrm{thick}_{R\underline{a}}(R\underline{a}). \end{aligned}$$

□

4.6. Assume that $\mathrm{char}(k) = p > 0$ and let $\underline{a} \in \mathbb{A}^r$. We have the following commutative diagram:

$$\begin{tikzcd} k[a_1 z_1 + \cdots + a_r z_r] \arrow[r, "\subseteq", "\text{polynomial extension}"'] \arrow[d, two heads] & k[z_1, \ldots, z_r] \arrow[d, two heads] \\ \frac{k[a_1 z_1 + \cdots + a_r z_r]}{(a_1 z_1 + \cdots + a_r z_r)^p} \arrow[r, "\subseteq", "\text{polynomial extension}"'] & \frac{k[z_1, \ldots, z_r]}{(a_1^p z_1^p + \cdots + a_r^p z_r^p)} \end{tikzcd}$$

Note that $a_1^p z_1^p + \cdots + a_r^p z_r^p = (a_1 z_1 + \cdots + a_r z_r)^p$ due to $p > 0$. Moreover, observe that algebras of the form

$$\frac{k[a_1 z_1 + \cdots + a_r z_r]}{(a_1 z_1 + \cdots + a_r z_r)^p}, \quad \underline{a} \in \mathbb{A}^r,$$

are precisely those that occure as group algebras of cyclic shifted subgroups of E defined by some given $\underline{a} \in \mathbb{A}^r$.

4.7. Consider the following *fact*: If R is a commutative ring and $M \in \mathcal{D}^f(R[\underline{t}])$ such that $H^*(M)$ is finitely generated over R, then

$$M \in \mathrm{thick}_{R[\underline{t}]}(R[\underline{t}]) \quad \iff \quad M \downarrow_R \in \mathrm{thick}_R(R).$$

Thus,

$$\begin{aligned} \underline{a}^p := (a_1^p, \ldots, a_r^p) \notin V_R(M) \quad &\iff \quad M \downarrow_{\frac{k[\sum_i z_i]}{(\sum_i a_i z_i)^p}} \text{ is perfect} \\ &\iff \quad M \downarrow_{\frac{k[\sum_i z_i]}{(\sum_i a_i z_i)^p}} \text{ is projective (for } M \text{ a module)} \end{aligned}$$

meaning that we have recovered Dade's Lemma.