Elementary subalgebras of Lie algebras

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Outline

I. The variety $\mathbb{E}(r,\mathfrak{g})$ of elementary subalgebras. (No representation theory here, just structure of Lie algebras.)

II. Local (r,j)-radical rank of a $\mathfrak{u}(\mathfrak{g})$-module M,

$$\epsilon \in \mathbb{E}(r,\mathfrak{g}) \mapsto \dim(Rad^j(\epsilon^*(M))).$$

III. Coherent sheaves on $X \subset \mathbb{E}(r,\mathfrak{g})$ associated to M.

IV. Algebraic vector bundles.

V. Some open problems.

Precedents

1.) Quillen's fundamental papers in the 1970's on the spectrum of equivariant cohomology rings.

• Remarkable qualitative information about the cohomology algebra $H^*(G,k)$.

• Recognition of the critical role played by elementary abelian subgroups $E \simeq \mathbb{Z}/p^{\times r} \subset G$.

2.) Papers with Brian Parshall in the 1980's on cohomology and support varieties for a restricted Lie algebra $\mathfrak{g}$.

• Spectrum of cohomology of $H^*(\mathfrak{u}(\mathfrak{g}),k)$ is geometrically richer than that for finite groups.

• Nilpotent subalgebras $\mathfrak{u} \subset \mathfrak{g}$ are an imperfect analogue of $E \subset G$.

• Comparison of rank varieties and cohomological varieties as for finite groups (generalized Carlson Conjecture).

Precedents (cont.)

3.) Papers with Chris Bendel and Andrei Suslin in the 1990's on 1-parameter subgroups for infinitesimal group schemes.

- Introduced 1-parameter subgroups to analyze cohomology of Frobenius kernels of algebraic groups.
- Precise role of the p-nilpotent cone for height 1 (i.e., restricted Lie algebras).

4.) Papers with Julia in the 2000's on π-points and constructions for infinitesimal group schemes.

- Encompasses finite groups, restricted Lie algebras, Frobenius kernels, etc.
- Suppresses cohomology (lurking in the background).
- Generalized support varieties.
- Constructions of vector bundles.

Precedents (cont.)

5.) Representations of elementary abelian p-groups and bundles on Grassmannians with Jon & Julia (2012).

• Introduces study of “shifted subgroups of rank $r > 1$” for an elementary abelian p group of some rank $n > r$.

• Gives various examples showing that get new information about representations if consider $r > 1$.

• Get bundles on Grassmannians (extending construction of bundles on projective spaces as Dave Benson discussed).

We extend (5.) to arbitrary restricted Lie algebras, as we are about to see.

Conventions

- k is an algebraically closed field (for convenience)
- k has characteristic $p > 0$ (there is a characteristic 0 analogue)
- $\mathfrak{g}$ is a finite dimensional Lie algebra over k
- $\mathfrak{g}$ is equipped with a p-operator $(-)^{[p]} : \mathfrak{g} \to \mathfrak{g}$ (i.e., $\mathfrak{g}$ is a restricted Lie algebra)
- $\mathfrak{u}(\mathfrak{g})$ is the restricted enveloping algebra of $\mathfrak{g}$, a cocommutative Hopf algebra over k of dimension $p^{\dim \mathfrak{g}}$
- M will denote a finite dimensional $\mathfrak{u}(\mathfrak{g})$-module (methods apply to other fg-representations)

Examples of restricted Lie algebras

$\mathfrak{g} \simeq \mathfrak{g}_a^{\oplus n}$; equivalently $\mathfrak{g}$ is abelian with $(-)^{[p]} = 0$.

MAIN EXAMPLE:
$\mathfrak{g} = Lie(G)$, for any connected algebraic group G over k.

Especially interesting, $\mathfrak{g} = Lie(G)$ where G is a simple algebraic group. Also interesting to consider Lie algebras of parabolic and unipotent subgroups of simple algebra groups.

$Der(k[t_1, \ldots, t_n]/(t_1^p, \ldots, t_n^p))$.

$\widetilde{\mathfrak{gl}}_n \simeq k \oplus \mathfrak{gl}_n$ as Lie algebras, and $(-)^{[p]}(a, X) = (\chi(X), 0)$ for some semi-linear character $\chi : \mathfrak{gl}_n \to k$.

Definitions

Definition

A restricted Lie subalebra $\epsilon \subset \mathfrak{g}$ is said to be an elementary subalgebra of dimension r if $\epsilon \simeq \mathfrak{g}_a^{\oplus r}$.

Definition

The projective variety $\mathbb{E}(r, \mathfrak{g})$ is the closed subvariety of Grass$(r, \mathfrak{g})$ of those r-planes $\epsilon \subset \mathfrak{g}$ which are elementary subalgebras.

Remark: If $\mathfrak{g} = Lie(G)$ for some algebraic group G, then there is a natural action of G on $\mathbb{E}(r, \mathfrak{g})$ induced by the adjoint action of G on $\mathfrak{g}$.

Examples

- $\mathfrak{g} = \mathfrak{g}_a^{\oplus n}, \quad \mathbb{E}(r, \mathfrak{g}) = \mathrm{Grass}_{n,r}$
- $\mathbb{E}(1, \mathfrak{g}) \;\simeq\; \mathrm{Proj}(k[\mathcal{N}_p(\mathfrak{g})])$, where $\mathcal{N}_p(\mathfrak{g})$ is p-nilpotent cone .
- Premet: $\mathbb{E}(2, \mathfrak{gl}_n)$ is irreducible for $p > n > 2$.
- $\mathbb{E}(n^2, \mathfrak{gl}_{2n}) = \mathrm{Grass}_{2n,n}$.
- $\mathbb{E}(n(n+1), \mathfrak{gl}_{2n+1}) = \mathrm{Grass}_{2n+1,n} \coprod \mathrm{Grass}_{2n+1,n}$ for $n > 1$.
- $\mathbb{E}(\frac{n(n+1)}{2}, \mathfrak{sp}_{2n})$ is the variety of Lagrangian n-planes $LG(n, V_{2n})$ in $V_{2n} \simeq k^{\oplus 2n}$.
- $\mathbb{E}(n^2, \widetilde{\mathfrak{gl}}_n)$ is a hypersurface in $\mathrm{Grass}_{2n,n}$.

Problems

1. Give a cohomological interpretation of $R = max\{r : \mathbb{E}(r, \mathfrak{g}) \neq \emptyset\}$.
2. Determine $\mathbb{E}(R, Lie(G))$ for all simple groups G.
3. Give sufficient conditions on $(\mathfrak{g}, r)$ to insure that $\mathbb{E}(r, \mathfrak{g})$ is irreducible.
4. Give sufficient conditions on $(\mathfrak{g}, r)$ to insure that $\mathbb{E}(r, Lie(G))$ a finite union of G-orbits.

GENERAL PROBLEM: Using the representation theory of $\mathfrak{g}$, describe $\mathbb{E}(r, \mathfrak{g})$ as a (Zariski) topological space.

Definitions

BASIC PRINCIPLE: Investigate $\mathfrak{u}(\mathfrak{g})$-module M by considering the $\mathfrak{u}(\epsilon) \simeq k(\mathbb{Z}/p^{\times r})$-modules $\epsilon^*(M)$ for $\epsilon \in \mathbb{E}(r, \mathfrak{g})$ for a given $r \geq 1$.

Definition

For any $\epsilon \in \mathbb{E}_r(\mathfrak{g})$ with basis $\{u_1, \ldots, u_r\}$ and any $j,\ 1 \leq j < p-1$

$$\mathrm{Rad}^j(\epsilon^*(M)) = \sum_{j_1+\cdots+j_r=j} \mathrm{Im}\{u_1^{j_1} \cdots u_r^{j_r} : M \to M\}$$

$$\mathrm{Soc}^j(\epsilon^*(M)) = \bigcap_{j_1+\cdots+j_r=j} \mathrm{Im}\{u_1^{j_1} \cdots u_r^{j_r} : M \to M\}.$$

Remarks about radicals and socles

- For $r = 1$, the data of the dimensions of $\mathrm{Rad}^j(\epsilon^* M)$ for all j is equivalent to the isomorphism type of $\epsilon^* M$. (Analogous statement for socles.)
- For ϵ an elementary subalgebra of dimension $r \geq 2$, the representation type of $\mathfrak{u}(\epsilon)$ is wild (except $p = 2, r = 2$).
- Socle dimensions and radical dimensions give different information, especially for $r > 1$.
- $\mathrm{Soc}^j(\epsilon^*(M^\#))^\perp \;=\; \mathrm{Rad}^j(\epsilon^*(M))$.
- Examples show that data for a particular r does not determine data for $r' \neq r$.

Local radical and socle ranks

The local (r, j)-radical rank (respectively, the local (r, j)-socle rank) of M:

$$\epsilon \in \mathbb{E}(r, \mathfrak{g}) \;\mapsto\; \dim \mathrm{Rad}^j(\epsilon^*(M)),$$

$$\epsilon \in \mathbb{E}(r, \mathfrak{g}) \;\mapsto\; \dim \mathrm{Soc}^j(\epsilon^*(M)).$$

Proposition

$\epsilon \in \mathbb{E}(r, \mathfrak{g}) \;\mapsto\; \dim(\mathrm{Rad}^j(\epsilon^*(M))$ *is lower semi-continuous.*

$\epsilon \in \mathbb{E}(r, \mathfrak{g}) \;\mapsto\; \dim(\mathrm{Soc}^j(\epsilon^*(M))$ *is upper semi-continuous.*

If $\mathfrak{g} = Lie(G)$, then both functions are constant on G-orbits.

Proof uses the “vectorial Casimir operator”.

r-rank varieties

Rank varieties, generalized to $r \geq 1$:

Definition

$\mathbb{E}(r, \mathfrak{g})_M \equiv \{\epsilon \in \mathbb{E}(r, \mathfrak{g}) : \epsilon^*(M) \text{ is not free}\}$.

Observation: $\mathbb{E}(r, \mathfrak{g})_M$ is a closed subvariety of the projective variety $\mathbb{E}(r, \mathfrak{g})$.

Remark: This can be computed in terms of ordinary rank varieties . . . newly packaged invariants, but not particularly new.

Generalized support varieties

The following construction extends for $r = 1$ to $r \geq 1$ the generalized support varieties of F-Pevtsova. These are new geometric invariants for $\mathfrak{u}(\mathfrak{g})$-modules.

Proposition

For any finite dimensional $\mathfrak{u}(\mathfrak{g})$-module M and any j, $1 \leq j < p$, the following are closed subvarieties of $\mathbb{E}(r, \mathfrak{g})$:

$$\mathbb{R}ad^j(r, \mathfrak{g})_M \equiv \{\epsilon \in \mathbb{E}(r, \mathfrak{g}) : \dim(\mathrm{Rad}^j(\epsilon^* M)) \text{ is not maximal}\}$$

$$\mathbb{S}oc^j(r, \mathfrak{g})_M \equiv \{\epsilon \in \mathbb{E}(r, \mathfrak{g}) : \dim(\mathrm{Soc}^j(\epsilon^* M)) \text{ is not minimal}\}.$$

Modules of constant (r, j)-radical (socle) rank

Observation: $\mathbb{R}ad^j(r, \mathfrak{g})_M = \emptyset$ iff $\dim(\mathrm{Rad}^j(\epsilon^* M))$ is constant.

Similarly, $\mathbb{S}oc^j(r, \mathfrak{g})_M = \emptyset$ iff $\dim(\mathrm{Soc}^j(\epsilon^* M))$ is constant.

Such modules are natural analogues of modules of "constant j-rank" for $r = 1$. (Analogues of module of constant Jordan type are modules M of either constant (r, j)-radical rank or constant (r, j)-socle rank for all j and a given r.)

Such modules lead to vector bundles on $\mathbb{E}(r, \mathfrak{g})$.

Challenge Find interesting $\mathfrak{u}(\mathfrak{g})$-modules (ones without many symmetries) which have constant (r, j)-radical or (r, j)-socle rank for various r, j.

Coherent sheaves

Theorem

For any (finite dimensional) $\mathfrak{u}(\mathfrak{g})$-module M, any r, any j, there are naturally defined coherent sheaves

$$\mathcal{I}m^j(M), \quad \mathcal{K}er^j(M) \quad on \quad \mathbb{E}(r, \mathfrak{g})$$

whose fibers over a sufficiently general $\epsilon \in \mathbb{E}(r, \mathfrak{g})$ are $\mathrm{Rad}^j(\epsilon^*(M)), \mathrm{Soc}^j(\epsilon^*(M))$.

Moreover, if $\mathfrak{g} = Lie(G)$ and M is a rational G-module, then $\mathcal{I}m^j(M)$, $\mathcal{K}er^j(M)$ are G-equivariant sheaves on $\mathbb{E}(r, \mathfrak{g})$.

vectorial Casimir operator

Proof of theorem involves a matrix form of the " Casimir operator" which in turn is the natural extension of the operator one uses for elementary abelian p-groups. For each s, $1 \leq s \leq r$, we consider

$$\Theta_s : M \otimes k[\mathcal{N}_p^r(\mathfrak{g})] \to M \otimes k[\mathcal{N}_p^r(\mathfrak{g})], \quad m \otimes f \mapsto \sum_{i=1}^{n} x_i m \otimes Y_{i,s} f$$

where $\mathcal{N}_p^r(\mathfrak{g}) \subset \mathfrak{g}^{\times r}$ is the variety of r-tuples on pairwise commuting nilpotent elements of $\mathfrak{g}$, $\{x_1, \ldots x_n\}$ is a basis for $\mathfrak{g}$, and $Y_{i,s}$ is a matrix function on $\mathfrak{g}$.

Proposition

The image, kernel, and cockerel sheaves do not depend upon the choice of basis of $\mathfrak{g}$.

General $X \subset \mathbb{E}(r, \mathfrak{g})$

Theorem

The construction of image and kernel sheaves generalizes to yield coherent sheaves

$$\mathcal{I}m^{j,X}(M), \quad \mathcal{K}er^{j,X}(M)$$

on any locally closed subvariety $X \subset \mathbb{E}(r, \mathfrak{g})$. Moreover,

$$\mathcal{I}m^{j,X}(M)_\epsilon \simeq \mathrm{Rad}^j(\epsilon^*(M)),$$
$$\mathcal{K}er^{j,X}(M)_\epsilon \simeq \mathrm{Soc}^j(\epsilon^*(M))$$

for all ϵ in some dense open subset of X.

Construction involves restriction of the generalized Casimir operators; sheaves are *not* restrictions of the global sheaves.

Vector bundles

Proposition

Let M be a $\mathfrak{u}(\mathfrak{g})$-module of constant (r, j)-radical rank (respectively, constant (r, j)-socle rank). Then $\mathcal{I}m^j(M)$, (resp. $\mathcal{K}er^j(M)$) is a vector bundle on $\mathbb{E}(r, \mathfrak{g})$.

Proposition

Let $\mathfrak{g} = Lie(G)$, M a rational G-module, $X = G \cdot \epsilon \subset \mathbb{E}(r, \mathfrak{g})$, and $P \subset G$ the stabilizer of $\epsilon \in \mathbb{E}(r, \mathfrak{g})$. Then $\mathcal{I}m^{j,X}(M)$, $\mathcal{K}er^{j,X}(M)$ are the algebraic vector bundles on X identified as

$$\mathcal{I}m^{j,X}(M) \simeq \mathcal{L}_{G/P}(Rad^j(\epsilon^* M)),$$

$$\mathcal{K}er^{j,X}(M) \simeq \mathcal{L}_{G/P}(\mathrm{Soc}^j(\epsilon^* M)).$$

Examples

Example

- Let $\mathfrak{g} = \mathfrak{gl}_{2n}$, $r = n^2$. Then
$$\mathcal{I}m^1(V_{2n}) \simeq \gamma_n, \quad \text{for} \quad M = V_{2n},$$
where γ_n is the canonical subbundle on $\mathrm{Grass}_{2n,n}$. Similar statement for $\mathfrak{g} = sp_{2n}$.

- Let $\mathfrak{g} = \mathfrak{gl}_n$, $\epsilon = \mathfrak{u}_{r,n-r} \in \mathbb{E}(r, \mathfrak{gl}_n)$, $X = GL_n \cdot \epsilon$, $m \leq n - r$. Then
$$\mathcal{I}m^{m,X}(M) = \gamma_r^{\otimes m}, \quad \text{for} \quad M = V_n^{\otimes m}.$$
Similar statements for $M = S^m(V_n)$, $M = \Lambda^m(V_n)$.

Examples (cont.)

Example

- Let G be a simple algebraic group, $P \subset G$ a cominiscule parabolic subgroup, $X = G/P$. Then

$$Coker^{G/P}(\mathfrak{g}) \simeq \mathbb{T}_{G/P}.$$

  (For example, $G = SL_n$, $P = P_{r,n-r}$ so that $G/P \simeq Grass_{r,n}$.)

- Let $G = Sp_{2n}$ and $P \subset G$ cominiscule parabolic. Assume that $p > 3$ does not divide $n+1$. Then

$$\mathcal{I}m^2(\mathfrak{g}) \simeq S^2(\gamma_n), \quad \text{on} \quad LG_{2n,n} = \mathbb{E}\left(\frac{n+1}{2}, \mathfrak{g}\right).$$

Semi-direct products

Let H be an algebraic group, W a faithful rational representation, and set

$$G_{W,H} \;=\; \mathbb{W}_{(1)} \rtimes H, \quad \mathfrak{g}_{W,H} = Lie(G_{W,H}).$$

Proposition

Let $\epsilon \subset W$ be an r-dimensional subspace viewed as $\epsilon \in \mathbb{E}(r, \mathfrak{g}_{W,H})$ and let $Y = H \cdot \epsilon$. Then for some $G_{W,H}$-module M, we have an isomorphism of vector bundles on Y,

$$\mathcal{I}m^{j,Y}(M) \simeq S^j(\gamma_r)_{|Y}, \quad 1 \leq j \leq p-1$$

where γ_r is the canonical subbundle of $\mathrm{Grass}(r, W)$.

Note: Y is a very general homogeneous space.

Questions

1.) How to relate to cohomology?

2.) What about more general finite group schemes?

3.) Many realization questions (of vector bundles, of radical and socle ranks, of subvarieties).

4.) Explore elementary subalgebras for $\mathfrak{g}$ in characteristic 0.

5.) Relate $K_0(\mathfrak{u}(\mathfrak{g}))$ to $K_0(\mathbb{E}(r, \mathfrak{g}))$.

6.) Exhibit *interesting, "new"* vector bundles.