Generic kernels and other constructions

Jon F. Carlson
University of Georgia

August 3, 2012

Cohomology and Supports, University of Washington

with Eric Friedlander and Julia Pevtsova

Ground Rules

G is a finite group (scheme)

and k is a field of characteristic p.

We assume that k is algebraically closed.

Definition

Given a finite group scheme G, a π-point is a flat map of K-algebras $\alpha_K : K[t]/t^p \longrightarrow KG$, for some field extension K/k, that factors by flat maps through the group algebra $KU_K \subset KG_K$ of some unipotent abelian subgroup scheme $U_K \subset G_K$.

A p-point is one that does not involve the field extension. That is, for a p-point α, $K = k$, which is algebraically closed.

Definition

Given a finite group scheme G, a π-point is a flat map of K-algebras $\alpha_K : K[t]/t^p \longrightarrow KG$, for some field extension K/k, that factors by flat maps through the group algebra $KU_K \subset KG_K$ of some unipotent abelian subgroup scheme $U_K \subset G_K$.

A p-point is one that does not involve the field extension. That is, for a p-point α, $K = k$, which is algebraically closed.
In general the collection of π-points is too large and so we define an equivalence.

Definition

Two π-points α_K, β_L are said to be equivalent if for every finitely generated kG-module M, the $K[t]/t^p$-module $\alpha_K^*(M_K)$ is projective if and only if the $L[t]/t^p$-module $\beta_L^*(M_L)$ is projective.

Theorem

(Friedlander-Pevtsova) For G a finite group scheme, the scheme $\Pi(G)$, of all equivalence classes of π-points, is homeomorphic to the scheme Proj $\mathrm{H}^*(G, k)$, *the projective prime ideal spectrum of* $\mathrm{H}^*(G, k)$

Theorem

(Friedlander-Pevtsova) For G a finite group scheme, the scheme $\Pi(G)$, of all equivalence classes of π-points, is homeomorphic to the scheme Proj $\mathrm{H}^*(G,k)$*, the projective prime ideal spectrum of* $\mathrm{H}^*(G,k)$

Definition

A π^r-point is a homomorphism $\alpha_K : K[t_1,\dots,t_r]/(t_1^p,\dots,t_r^p) \to KG$ that factors by flat maps through some unipotent abelian subgroup scheme, for some extension K of k.

We have an equivalence relation of the same sort. If $G = E \simeq (\mathbb{Z}/p\mathbb{Z})^n$, then the set of equivalence classes of π^r-points is homoemorphic to $\mathrm{Grass}_{r,n}$ the Grassmannian of r-planes in n-space.

Suppose that E is an elementary abelian p-group of rank n. Then

$$kE \;\cong\; k[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p).$$

(If $E = \langle g_1, \ldots, g_n \rangle$ then let $x_i = g_i - 1$.)

A map $\alpha : K[t]/(t^p) \to KE$ is a π-point if and only if $\alpha(t) \notin \mathrm{Rad}^2(KE)$.

This means that $\alpha(t) = \sum a_i x_i + w$ where not all a_i's are zero and w is a sum of monomials of degree at least 2, in the generators $x_1, \ldots, x_n$.

If x is a nilpotent operator on a vector space V, then the isomorphism class of the action of x is characterized by the Jordan type of x - the Jordan canonical form of x. Because x is nilpotent, all of its eigenvalues are zero. Hence the only thing of importance is the size of the Jordan blocks. The Jordan type is the list of sizes of the Jordan block and it is a partition of the dimension n of V.

If x is a nilpotent operator on a vector space V, then the isomorphism class of the action of x is characterized by the Jordan type of x - the Jordan canonical form of x. Because x is nilpotent, all of its eigenvalues are zero. Hence the only thing of importance is the size of the Jordan blocks. The Jordan type is the list of sizes of the Jordan block and it is a partition of the dimension n of V.

If M is a module over kG, then we say that M has constant Jordan type if $\alpha_K(t)$ has the same Jordan type for every π-point $\alpha_K : K[t]/(t_p) \longrightarrow KG$.

Jordan Types

If x is a nilpotent operator on a vector space V, then the isomorphism class of the action of x is characterized by the Jordan type of x - the Jordan canonical form of x. Because x is nilpotent, all of its eigenvalues are zero. Hence the only thing of importance is the size of the Jordan blocks. The Jordan type is the list of sizes of the Jordan block and it is a partition of the dimension n of V.

If M is a module over kG, then we say that M has constant Jordan type if $\alpha_K(t)$ has the same Jordan type for every π-point $\alpha_K : K[t]/(t_p) \longrightarrow KG$.

Modules of constant Jordan type are closed under direct sums, tensor products, and direct summands.

Thought for the day

If you can't prove something,

If you can't prove something,

then maybe

If you can't prove something,

then maybe

it's NOT true!

Example

If $E = \langle x, y \rangle \cong C_p^2$ (elementary abelian of rank 2), and M is a kE-module with constant Jordan type, then M has a distinguished submodule $\mathfrak{K}(M)$ and a filtration

$$0 \subseteq x^{p-1}\mathfrak{K}(M) \subseteq \cdots \subseteq x\mathfrak{K}(M) \subseteq \mathfrak{K}(M)$$

$$\subseteq x^{-1}\mathfrak{K}(M) \subseteq \cdots \subseteq x^{-p+1}\mathfrak{K}(M) = M$$

which does not depend on x (can substitute $ax + by$ for x and get exactly the same filtration). The modules $x^i\mathfrak{K}(M)$ all have constant Jordan type for $i \geq 0$.

Question(CFS): Do the modules $x^{-i}\mathfrak{K}(M)$ have constant Jordan type?

Example

If $E = \langle x, y \rangle \cong C_p^2$ (elementary abelian of rank 2), and M is a kE-module with constant Jordan type, then M has a distinguished submodule $\mathfrak{K}(M)$ and a filtration

$$0 \subseteq x^{p-1}\mathfrak{K}(M) \subseteq \cdots \subseteq x\mathfrak{K}(M) \subseteq \mathfrak{K}(M)$$
$$\subseteq x^{-1}\mathfrak{K}(M) \subseteq \cdots \subseteq x^{-p+1}\mathfrak{K}(M) = M$$

which does not depend on x (can substitute $ax + by$ for x and get exactly the same filtration). The modules $x^i\mathfrak{K}(M)$ all have constant Jordan type for $i \geq 0$.

Question(CFS): Do the modules $x^{-i}\mathfrak{K}(M)$ have constant Jordan type?

Answer(Baland): NO!

The collection of modules of constant Jordan type is closed under tensor products (CFP).

Question: So how about arbitrary symmetric products? exterior products? Schur functors?

The collection of modules of constant Jordan type is closed under tensor products (CFP).

Question: So how about arbitrary symmetric products? exterior products? Schur functors?

Answer(Lim): NO!

Yet another Example

Theorem (CFS)

Let $E = \langle x, y \rangle \cong C_p^2$ be an elementary abelian p-group of rank 2. If M is a cyclic module having constant Jordan type, then $M \cong kE/\operatorname{Rad}^n(kE)$ for some value of n.

Question: Does the same hold for elementary abelian p-groups of larger rank?

Yet another Example

Theorem (CFS)

Let $E = \langle x, y \rangle \cong C_p^2$ be an elementary abelian p-group of rank 2. If M is a cyclic module having constant Jordan type, then $M \cong kE/\operatorname{Rad}^n(kE)$ for some value of n.

Question: Does the same hold for elementary abelian p-groups of larger rank?

Answer(CFP): NO!

Yet another Example

Theorem (CFS)

Let $E = \langle x, y \rangle \cong C_p^2$ be an elementary abelian p-group of rank 2. If M is a cyclic module having constant Jordan type, then $M \cong kE/\operatorname{Rad}^n(kE)$ for some value of n.

Question: Does the same hold for elementary abelian p-groups of larger rank?

Answer(CFP): NO!

What can you do with three (or more) variables that you can't do with two?

What can you do with three (or more) variables that you can't do with two?

$$ax^2 + bxy + cy^2 = a(x - \gamma_1 y)(x - \gamma_2 y)$$

What can you do with three (or more) variables that you can't do with two?

$$ax^2 + bxy + cy^2 = a(x - \gamma_1 y)(x - \gamma_2 y)$$

But $\qquad x^2 - yz \qquad$ is irreducible.

What can you do with three (or more) variables that you can't do with two?

$$ax^2 + bxy + cy^2 = a(x - \gamma_1 y)(x - \gamma_2 y)$$

But $\quad x^2 - yz \quad$ is irreducible.

Let $E = C_p^n$, $n \geq 3$. Define M by

$$0 \longrightarrow k \xrightarrow{\mu} kE/\operatorname{Rad}^3(kE) \longrightarrow M \longrightarrow 0$$

where $\mu(1)$ is a homogeneous irreducible polynomial of degree 2 in the elements $x_1, \ldots, x_n$. Then M has constant Jordan type. Also there is an infinite collection of mutually nonisomorphic such modules.

What can you do with three (or more) variables that you can't do with two?

$$ax^2 + bxy + cy^2 = a(x - \gamma_1 y)(x - \gamma_2 y)$$

But $\quad x^2 - yz \quad$ is irreducible.

Let $E = C_p^n$, $n \geq 3$. Define M by

$$0 \longrightarrow k \xrightarrow{\mu} kE/\operatorname{Rad}^3(kE) \longrightarrow M \longrightarrow 0$$

where $\mu(1)$ is a homogeneous irreducible polynomial of degree 2 in the elements $x_1, \dots, x_n$. Then M has constant Jordan type. Also there is an infinite collection of mutually nonisomorphic such modules.

Question What are the bundles associated to these modules?

Suppose that E is an elementary abelian p-group of rank n. Then

$$kE \;\cong\; k[x_1, \ldots, x_n]/(t_1^p, \ldots, t_n^p).$$

Let $\mathbb{V} = k\text{-Span}\{x_1, \ldots, x_n\} \subseteq kE$

We say that a kE-module M has the equal r-images property if for any subspace W of $\mathbb{V}$ of dimension r, we have that $\sum_{w \in W} w \cdot M = \mathrm{Rad}(M) = \sum_{v \in \mathbb{V}} v \cdot M$.

Note that if R_W is the unital subalgebra of kE generated by W, then $\sum_{w \in W} w \cdot M = \mathrm{Rad}(R_W)M$.

For W a subspace of dimension r in $\mathbb{V}$, let R_W be the unital subalgebra of kE generated by W.

A kE-module M has the equal r-images property if for any subspace W of $\mathbb{V}$ of dimension r, we have that $\mathrm{Rad}(R_W)M = \mathrm{Rad}(M)$.

A kE-module M has the equal r-kernels property if for any subspace W of $\mathbb{V}$ of dimension r, we have that $\mathrm{Soc}(R_W)M = \mathrm{Soc}(M)$.

Any quotient of a module with the equal r-images property has the equal r-images property.

If L_1 and L_2 are submodules having the equal r-images property of a module M, then so also is $L_1 + L_2$.

If M has the equal r-images property, then so does $\mathrm{Rad}^i(M)$ for any i.

For any kE-module M, let $\mathcal{M}_r(M)$ be the largest submodule with the equal r-images property. Then $\mathcal{M}_r$ is a functor which is independent of the choice of the generators $\mathbb{V}$ of kE.

Assume E is an elementary abelian p-group of rank r. Suppose that M is a kE-module. For S an open subset of $\mathrm{Grass}_r(\mathbb{V})$, let

$$ {}_S M \ = \ \sum_{W \in S} \{ m \mid \quad wm = \{0\} \quad \text{for all } w \in W \}. $$

The *generic r-kernel* of a kE-module M is the intersection

$$ \mathfrak{K}_r(M) \ = \ \bigcap_{S} \quad {}_S M $$

where the intersection is taken over all open sets S in $\mathrm{Grass}_r(\mathbb{V})$.

Note, for any M, $\mathfrak{K}_r(M) = {}_S M$ for some S. Also, $\mathfrak{K}_r(\mathfrak{K}_r(M)) = \mathfrak{K}_r(M)$, $\mathfrak{K}_r(M) \subseteq \mathfrak{K}_{r-1}(M)$.

$\mathfrak{K}_r$ is a functor.

We can prove that the generic r-kernel has the equal $(n-r)$-images property. That is, for any M, $\mathfrak{K}_r(M) \subseteq \mathcal{M}_{n-r}(M)$.

Also, in the case that $n = 2$ and $r = 1$, we know that $\mathfrak{K}_1 = \mathcal{M}_1$ (CFS).

We can prove that the generic r-kernel has the equal $(n-r)$-images property. That is, for any M, $\mathfrak{K}_r(M) \subseteq \mathcal{M}_{n-r}(M)$.

Also, in the case that $n = 2$ and $r = 1$, we know that $\mathfrak{K}_1 = \mathcal{M}_1$ (CFS).

Question: Is $\mathfrak{K}_r = \mathcal{M}_{n-r}$, in general?

We can prove that the generic r-kernel has the equal $(n-r)$-images property. That is, for any M, $\mathfrak{K}_r(M) \subseteq \mathcal{M}_{n-r}(M)$.

Also, in the case that $n = 2$ and $r = 1$, we know that $\mathfrak{K}_1 = \mathcal{M}_1$ (CFS).

Question: Is $\mathfrak{K}_r = \mathcal{M}_{n-r}$, in general?

Answer: NO!

We can prove that the generic r-kernel has the equal $(n-r)$-images property. That is, for any M, $\mathfrak{K}_r(M) \subseteq \mathcal{M}_{n-r}(M)$.

Also, in the case that $n = 2$ and $r = 1$, we know that $\mathfrak{K}_1 = \mathcal{M}_1$ (CFS).

Question: Is $\mathfrak{K}_r = \mathcal{M}_{n-r}$, in general?

Answer: NO!

There exist a module with the equal 1-images property, but with $\mathfrak{K}_{n-1}(M) \neq M = \mathcal{M}_1(M)$.

What is generic about the generic kernel?

Let M be a kE-module of dimension d. Let $\mathcal{V}_{r,d}$ be the variety of representations of dimension d of an elementary abelian group of rank r. Then there is an algebraic map

$$\psi_M : \mathsf{Grass}_r(\mathbb{V}) \to \mathcal{V}_{r,d}/\mathsf{GL}(r,k)$$

that takes an r-plane W to the class of the representation of the restriction to R_W.

Any closed set in $\mathcal{V}_{r,d}$ pulls back to a closed set in $\mathsf{Grass}_r(\mathbb{V})$ which is avoided in the construction of the generic kernel. That is the generic kernel represents a construction of the generic point of the image of that map.

Example: Let $\mathfrak{U}$ be some collection of open sets in $\text{Grass}_r(\mathbb{V})$, as for example, all cofinite subsets. For S in $\mathfrak{U}$, let

$$ {}_S M \;=\; \sum_{W \in S} \{ m \mid \quad wm = \{0\} \quad \text{for all } w \in W \}. $$

The *generic r-kernel* of a kE-module M relative to $\mathfrak{U}$ is

$$ \mathfrak{K}_r(M) \;=\; \bigcap_{S \in \mathfrak{U}} \; {}_S M. $$

Example: Let $\mathfrak{U}$ be some collection of open sets in $\text{Grass}_r(\mathbb{V})$, as for example, all cofinite subsets. For S in $\mathfrak{U}$, let

$$_S M \;=\; \sum_{W \in S} \{m \mid \quad wm = \{0\} \quad \text{for all } w \in W\}.$$

The *generic r-kernel* of a kE-module M relative to $\mathfrak{U}$ is

$$\mathfrak{K}_r(M) \;=\; \bigcap_{S \in \mathfrak{U}} \; {_S M}.$$

These constructions seem to depend on the choice of generators $\mathbb{V}$?

Question: To what extent do these depend on the choice of $\mathbb{V}$?

Suppose that M is a kG-module. For S an open subset of $\mathbb{G}_r(\mathfrak{B})$, the set of equivalence classes of π^r-points in a component $\mathfrak{B}$ of the set $\Pi^r(G)$ of all π^r-points, let $\hat{S}$ the set of all π^r-points whose classes are in S, and let

$$_SM^{(i)} = \sum_{\alpha \in \hat{S}} \{m \mid \quad \alpha(t_1)^i m = \cdots = \alpha(t_n)^i m = 0\}$$

The *generic* (n,i)*-kernel* of a kG-module M corresponding to a component $\mathfrak{B}$ of $\Pi^r(G)$ is the intersection

$$\mathfrak{K}^{\mathfrak{B}}_{n,i}(M) = \bigcap_S \quad {}_SM^{(i)}$$

where the intersection is taken over all open sets S in $\mathbb{G}_n(\mathfrak{B})$.

"subidentity functor" = "subfunctor of the identity functor"

All of the functors we have discussed are examples of subidentity functors.

"subidentity functor" = "subfunctor of the identity functor"

All of the functors we have discussed are examples of subidentity functors.

Suppose that $N \subseteq M$ are kG-modules. Let $\mathfrak{D}_N^M$ be the subidentity functor given by

$$\mathfrak{D}_N^M(U) \quad = \quad \sum_{\varphi} \varphi(N)$$

where the sum is over all φ in $\mathrm{Hom}_{kG}(M, U)$.

Note that for any s,

$$\mathrm{Rad}^s = \mathfrak{D}^{kG}_{\mathrm{Rad}^s(kG)} \quad \text{and} \quad \mathrm{Soc}^s = \mathfrak{D}^{kG/\,\mathrm{Rad}^s(kG)}_{kG/\,\mathrm{Rad}^s(kG)}$$

Finitely generated?

Note that for any subidentity functor F, $F = \sum_M \mathfrak{D}^M_{F(M)}$.

Finitely generated?

Note that for any subidentity functor F, $F = \sum_M \mathfrak{D}^M_{F(M)}$.

So call a subidentity functor "finitely generated" if it has the form $\mathfrak{D}^M_N$ for a finitely generated module M.

Which functors are finitely generated?

Finitely generated?

Note that for any subidentity functor F, $F = \sum_M \mathfrak{D}^M_{F(M)}$.

So call a subidentity functor "finitely generated" if it has the form $\mathfrak{D}^M_N$ for a finitely generated module M.

Which functors are finitely generated?

If $n = 2$, $r = 1$ then $\mathcal{M}_1$ (largest submodule with equal images property) is not finitely generated.

Can find other nonfinitely generated subidentity functors based on support varieties.