PROBLEM SESSION

MODULES OF CONSTANT JORDAN TYPE AND VECTOR BUNDLES ON PROJECTIVE SPACE

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Throughout these questions, $E = (\mathbb{Z}/p)^r = \langle g_1, \dots, g_r \rangle$ is an elementary abelian p-group, k is a field of characteristic p and M is a finitely generated kE-module. We write X_i for the element $g_i - 1 \in kE$.

1. Modules of constant Jordan type

Question 1. Let $E = \mathbb{Z}/p \times \mathbb{Z}/p = \langle g_1, g_2 \rangle$ have rank two. Decide which of the following kE-modules have constant Jordan type.

a) $g_1 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad g_2 \mapsto \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \qquad (\lambda \in k).$

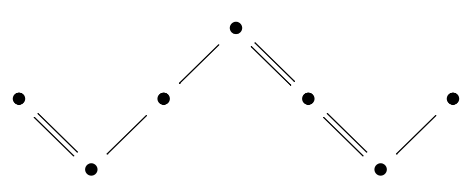
b) $g_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

c) $g_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad g_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

d) $g_1 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad g_2 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

[Hint: the answer depends on the characteristic of k]

e) ($p \geq 3$) The module with diagram



(Begin by writing down the matrices for this module)

f) The radical of the module in e).

Question 2. Which of the modules in question 1 have the constant image property?

Questions 3–7 are designed to show that there are a lot of modules of constant Jordan type for $\mathbb{Z}/p \times \mathbb{Z}/p$ for $p \geq 3$ and for $(\mathbb{Z}/p)^3$ for any prime.

Informally, an algebra A has *wild representation type* if we can define, for each pair of $n \times n$ matrices X and Y, a representation of A in such a way that X and Y can be recovered up to simultaneous conjugation.

Question 3. Show that for $r \geq 3$, kE has wild representation type, by considering the matrices

$$\begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \qquad \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \qquad \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}.$$

Question 4. By considering the matrices

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} X & I \\ I & Y \end{pmatrix},$$

show that classification of pairs of square matrices with no common eigenvectors, up to simultaneous conjugation, is of wild representation type.

Question 5. Consider the quiver

$$Q = \quad \bullet \underset{\delta}{\overset{\beta}{\rightrightarrows}} \bullet \underset{\gamma}{\overset{\alpha}{\rightrightarrows}} \bullet$$

with relation $\alpha\beta = \gamma\delta$. Use the diagram

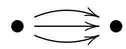
$$V \quad \underset{\left(\begin{smallmatrix} 0 \\ I \\ 0 \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} I \\ 0 \\ 0 \end{smallmatrix}\right)}{\rightrightarrows}} \quad V \oplus V \oplus V \quad \underset{\left(\begin{smallmatrix} X & I & 0 \\ Y & 0 & I \end{smallmatrix}\right)}{\overset{\left(\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \end{smallmatrix}\right)}{\rightrightarrows}} \quad V \oplus V$$

to show that this quiver has wild representation type.

Question 6. Show that in question 5 if X and Y have no common eigenvectors then for all λ and μ in k, not both zero, $\lambda\beta + \mu\delta$ is injective, $\lambda\alpha + \mu\gamma$ is surjective, and their composite is injective.

Use this to construct a wild set of modules of constant Jordan type for $\mathbb{Z}/p \times \mathbb{Z}/p$ when $p \geq 3$.

Question 7. Show that the quiver

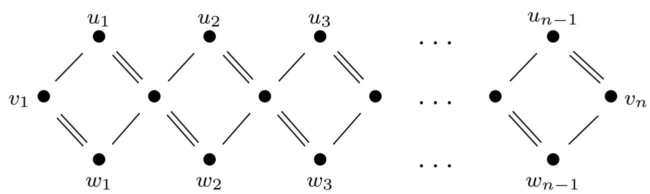


is of wild representation type by considering the matrices

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix} \qquad \begin{pmatrix} X & 0 & I & Y \\ 0 & 0 & 0 & I \end{pmatrix}$$

as maps from $V^{\oplus 4}$ to $V^{\oplus 2}$. Use this to construct a wild set of modules of constant Jordan type for $(\mathbb{Z}/p)^3$ for any prime p.

Question 8. Let M_n be the module with diagram



More explicitly, M_n has basis elements $u_1, \dots, u_{n-1}$, $v_1, \dots, v_n$ and $w_1, \dots, w_{n-1}$ with

$$X_1(u_i) = v_i \quad X_2(u_i) = v_{i+1} \quad X_1(v_{i+1}) = X_2(v_i) = w_i \quad (1 \le i \le n-1)$$

and all other basis elements sent to zero by X_1 and X_2. Show that M_n has constant Jordan type if and only if n is divisible by p, with Jordan type $[3]^{n-2}[2]^2$.

Question 9. Find the generic kernel of the module M_n given in question 8.
[Hint: use the fact that the generic kernel is the largest submodule with the constant image property.]

Question 10. Show that if M has constant Jordan type then so does $\Omega(M)$, the kernel of the projective cover of M. Is it also true that if M has the constant image property then so does $\Omega(M)$?

Question 11. If M has the constant image property, show that the image of each X_α^j $(0 \neq \alpha \in \mathbb{A}^r(k))$ is equal to $\mathsf{Rad}^j(M)$. Deduce that $\mathsf{Rad}^p(M) = 0$. What is the smallest value of n such that $\mathsf{Rad}^n(kE) = 0$?

Question 12. Let $E = \langle g \rangle$ be cyclic of order $p > 2$. If M is the indecomposable kE-module on which g acts with a Jordan block of length two, find the structure of $M \otimes M$, $\Lambda^2(M)$ and $S^2(M)$.

Question 13. Let $E = \langle g \rangle$ be cyclic of order p, and write J_i for the indecomposable kE-module on which g acts with a Jordan block of length i.

(1) Show that $J_2 \otimes J_i$ is isomorphic to $J_{i+1} \oplus J_{i-1}$ if $1 \le i \le p-1$ and to $J_p \oplus J_p$ if $i = p$.
(2) Find $J_3 \otimes J_3$ using the first part of the question and the associativity of tensor product. Treat the cases $p = 3$ and $p \ge 5$ separately.
(3) If $p \ge 5$ find $S^2(J_3)$ and $\Lambda^2(J_3)$.

Question 14. Let E have rank two, and let M be a kE-module of constant Jordan type. Let $\mathfrak{K}(M)$ be the generic kernel of M. Show that the following quantities for the subquotient $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ are independent of $0 \ne \alpha \in \mathbb{A}^2(k)$:

- The number of Jordan blocks of length one of X_α.
- The total number of Jordan blocks of X_α.
- The dimension of $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$.

Prove that $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ has constant Jordan type.

Question 15. Let $E = \mathbb{Z}/p \times \mathbb{Z}/p$. If M has constant Jordan type with no Jordan blocks of length one, it is known that the total number of Jordan blocks is divisible by p. Apply this to $\Omega(M)$ to deduce that if M has constant Jordan type with no Jordan blocks of length $p-1$ then the number of Jordan blocks of length p is divisible by p.

2. The stable module category

In preparation for working with vector bundles and modules of constant Jordan type, we begin with a set of exercises to get you used to the stable module category $\mathsf{stmod}(kE)$. Since the construction of the stable module category works just as well for any finite group G, we shall work in this context.

The stable module category $\mathsf{stmod}(kG)$ has the same objects as the module category $\mathsf{mod}(kG)$, but the morphisms are given by

$$\underline{\mathrm{Hom}}_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\mathrm{PHom}_{kG}(M,N)$$

where $\mathrm{PHom}_{kG}(M,N)$ is the linear subspace consisting of homomorphisms that factor through some projective (= injective) kG-module.

Question 16. (1) Show that the linear map $kG \to \mathrm{Hom}_k(kG,k)$ given by $g \mapsto (h \mapsto \delta_{g,h})$ is a kG-module isomorphism. Deduce that kG is an injective kG-module, and hence every projective kG-module is injective.

(2) If M is a kG-module, show that the kG-module $M{\downarrow_1}{\uparrow^G} = kG \otimes_k M$ (where $g \in G$ acts via $g(h \otimes m) = gh \otimes m$) is free, and hence projective.

(3) Show that the map $M \to M{\downarrow_1}{\uparrow^G}$ given by

$$m \mapsto \sum_{g \in G} g \otimes g^{-1} m$$

is an injective kG-module homomorphism. Thus every module embeds in a projective module. Deduce that every injective kG-module is projective.

Question 17. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of kG-modules. Let $P \to C$ be a projective module surjecting onto C with kernel $\Omega(C)$. Lift to a homomorphism $P \to B$ to obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(C) & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

to show that there is a short exact sequence

$$0 \to \Omega(C) \to P \oplus A \to B \to 0$$

in $\mathsf{mod}(kG)$.

Dually, embed A in an injective module I with cokernel $\Omega^{-1}(A)$ to obtain a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & I & \longrightarrow & \Omega^{-1}(A) & \longrightarrow & 0 \end{array}$$

and hence a short exact sequence

$$0 \to B \to I \oplus C \to \Omega^{-1}(A) \to 0$$

in $\mathsf{mod}(kG)$.

Question 18. We make $\mathsf{stmod}(kG)$ into a triangulated category in which the translation is the functor Ω^{-1}. The triangles are the triples of modules and homomorphisms

$$A \to B \to C \to \Omega^{-1}(A)$$

which are isomorphic to triples coming from short exact sequences

$$0 \to A \to B \to C \to 0$$

using the process described in Question 17 for obtaining the third map $C \to \Omega^{-1}(A)$.

If you have the stamina, check the axioms for a triangulated category. The third isomorphism theorem in $\mathsf{mod}(kG)$ will be required in order to verify the octahedral axiom for $\mathsf{stmod}(kG)$.

3. Vector bundles on projective space

Question 19. Consider the Euler sequence defining the tangent bundle

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus r} \to \mathcal{T} \to 0$$

where the first map in this sequence is given by the column vector $(Y_1, \dots, Y_r)^t$, and tensor with $\mathcal{O}(-1)$ to get

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus r} \to \mathcal{T}(-1) \to 0.$$

If $p = 2$, realise the first map in this sequence with a map

$$\Omega(k) \to k^{\oplus r}$$

and complete to a triangle in $\mathsf{stmod}(kE)$. Show that this gives a short exact sequence in $\mathsf{mod}(kE)$

$$0 \to \Omega(k) \to kE \oplus k^{\oplus r} \to M_{\mathcal{T}} \to 0.$$

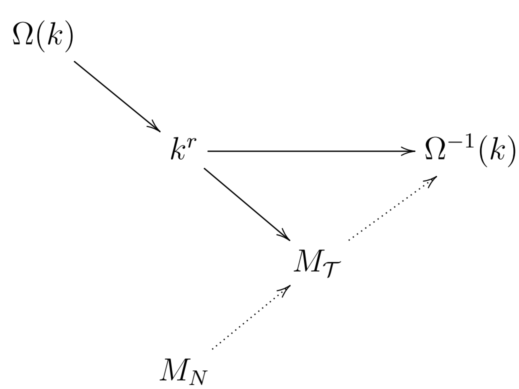
Write down matrices for the action of E on the $r + 1$ dimensional module $M_{\mathcal{T}}$.

Question 20. The *null correlation bundle* $\mathcal{F}_N$ on $\mathbb{P}^{r-1}$ (r even) is the homology in the middle place of the complex

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus r} \to \mathcal{O}(1) \to 0$$

where the first map is given by the column vector $(Y_1, \dots, Y_r)^t$ and the second map is given by $(Y_r, -Y_{r-1}, \dots, Y_2, -Y_1)$. If $p = 2$, construct a kE-module M_N of dimension $r + 2$ with $\mathcal{F}_1(M) \cong \mathcal{F}_N$. Write down matrices for the action of E on M_N.

Compare your construction with the following diagram in $\mathsf{stmod}(kE)$:



Question 21. For p odd and r even, give a construction in $\mathsf{stmod}(kE)$ of a module M of stable constant Jordan type $[1]^{r-2}$ with $\mathcal{F}_1(M) \cong F^*(\mathcal{F}_N)$, the Frobenius pullback of the null correlation bundle on $\mathbb{P}^{r-1}$.

Question 22. This question gives a simplified version of Tango's construction of rank $r-2$ vector bundles on $\mathbb{P}^{r-1}$ (not to be confused with the Tango bundle of rank 2 on $\mathbb{P}^5$ in characteristic two).

Let V be a vector space of dimension r over k, and let $V \times V \to \Lambda^2(V)$ be the map sending (x, y) to $x \wedge y$. Show that the image is a subvariety of dimension $2r-3$. Deduce that there is a linear subspace W of $\Lambda^2(V)$ of codimension $2r-3$ whose intersection with the image is just the origin. In other words, W contains no non-zero element of the form $x \wedge y$.

Now look at the beginning of the Koszul complex on $\mathbb{P}^{r-1}$, suitably twisted:

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus r} \to \mathcal{O}^{\binom{r}{2}} \to \mathcal{E} \to 0.$$

Thinking of $\mathcal{O}^{\binom{r}{2}}$ as $\Lambda^2(V) \otimes_k \mathcal{O}$, there is a trivial subsheaf $W \otimes_k \mathcal{O}$ that injects into $\mathcal{E}$ via the last map in the sequence. Define $\mathcal{F}_W$ to be the cokernel of $W \otimes_k \mathcal{O} \to \mathcal{E}$. Show that $\mathcal{F}_W$ is a vector bundle of rank $r-2$.

Question 23. For $p = 2$, construct a kE-module M with $J^3(M) = 0$ and with radical layers of dimensions 1, r, $2r-3$ such that $\mathcal{F}_1(M)$ is the vector bundle constructed in Question 22.

4. Chern classes

Throughout this section, let $R = k[Y_1, \ldots, Y_r]$, let M be a finitely generated graded R-module and let $\mathcal{F}$ be the resulting coherent sheaf on $\mathbb{P}^{r-1}$.

Question 24. Prove that the Poincaré series $p_M(t)$ takes the form

$$p_M(t) = \frac{f(t)}{(1-t)^r}$$

where $f(t)$ is a Laurent polynomial.
[Hint: Consider the kernel and cokernel of multiplication by Y_r on M and use induction on r.]

Question 25. Show that

$$\mathsf{Ch}(\mathcal{F}) = \mathrm{rank}(\mathcal{F}) + c_1 h + \tfrac{1}{2}(c_1^2 - 2c_2)h^2 + \cdots$$

and find the next term in this expansion.

Question 26. Show directly from the definition that

$$c(\mathcal{F}(1),h) = (1+h)^{\mathrm{rank}\mathcal{F}} c\left(\mathcal{F}, \frac{h}{1+h}\right).$$

Question 27. Show that the Chern polynomial of the null correlation bundle constructed in Question 20 is $1/(1-h^2)$.

Question 28. Show that the Chern polynomial of the vector bundles of Tango constructed in Question 22 is $(1-2h)/(1-h)^r$.

Question 29. Use congruences on Chern numbers to prove that if $r \geq 3$ and M has stable constant Jordan type $[2]\,[1]$ with $p \geq 5$ then $r = 3$ and $p \equiv 1 \pmod 3$. Find the possibilities for $c_1(\mathcal{F}_1(M))$ and $c_1(\mathcal{F}_2(M))$.

5. Hirzebruch–Riemann–Roch Theorem

Question 30. Prove that $s_n(\mathcal{F}) = \sum_j a_j j^n = \sum_j \alpha_j^n$. [Hint: take logs of both sides of the equation defining $c(\mathcal{F},h)$ and differentiate]

Question 31. Use Schwartzenberger's conditions to show:

(i) For a coherent sheaf on $\mathbb{P}^3$ we have $c_1c_2 + c_3 \equiv 0 \pmod 2$.

(ii) For a rank two vector bundle on $\mathbb{P}^4$ we have

$$c_2(c_2 + 1 - 3c_1 - 2c_1^2) \equiv 0 \pmod{12}.$$

Question 32. Let $p = 2$ and let M be a module of constant Jordan type $[2]^n$. Use the formula $c(\mathcal{F}_2(M))c(\mathcal{F}_2(M)(1)) = 1$ to prove that $n = -2c_1(\mathcal{F}_2(M))$. What can you deduce about $c_2(\mathcal{F}_2(M))$?

Question 33. Let $p = 2$ and $r = 4$ (i.e., $E \cong (\mathbb{Z}/2)^4$) and let M be a module of constant Jordan type $[2]^n$. Prove that

$$\mathcal{F}_2(M) = 1 - \frac{n}{2}h + \frac{n^2}{8}h^2 - \frac{n^3-4n}{48}h^3 \in \mathbb{Z}[h]/(h^4).$$

Without Hirzebruch–Riemann–Roch deduce that n is divisible by four. Using part (i) of the previous question, show that the Hirzebruch–Riemann–Roch theorem implies that n is divisible by eight.
[This also follows from Dade's lemma!]

Question 34. Use Poincaré series directly, instead of going through the Hirzebruch–Riemann–Roch theorem, to show that if M is a module of constant Jordan type $[2]^n$ then $2^{r-1}|n$.

Question 35. Use the Hirzebruch–Riemann–Roch theorem to prove that if M is a module of constant Jordan type $[2]^n[1]^2$ for $(\mathbb{Z}/2)^4$ then n is not congruent to 1, 3 or 5 modulo 8.

Question 36. We define a *nilvariety* of rank r and constant Jordan type $[a_1]\dots[a_t]$ to be a linear space of square matrices all non-zero elements of which have the same Jordan canonical form, with Jordan blocks of sizes $a_1,\dots,a_t$. The matrices do not necessarily commute, so they do not necessarily define a representation of $(\mathbb{Z}/p)^r$. Show that the matrices

$$\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\qquad\begin{pmatrix}0&0&0\\1&0&0\\0&-1&0\end{pmatrix}$$

span a nilvariety of rank two and constant Jordan type [3] in any characteristic.

Question 37. Show that a nilvariety of rank r and constant Jordan type $[2]^n$ is the same as a representation of an exterior algebra on r generators.

Question 38. Show, using Poincaré series, that a rank r nilvariety of constant Jordan type $[3]^n$ necessarily satisfies $3^{\lfloor\frac{r-1}{2}\rfloor}|n$. Use tensor products of the example from Question 36 to show that in characteristic three this is best possible.

Question 39 (Causa, Re, Teodorescu)**.** Show that if there is a nilvariety of rank r and constant Jordan type $[m]$ then $r\le 2$, and if $r=2$ then m is odd.
[Hint: If $r\ge 3$ then the line bundles $\mathcal{O}(n)$ do not extend each other: for all $n,n'\in\mathbb{Z}$ we have $\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^{r-1}}}(\mathcal{O}(n),\mathcal{O}(n'))=0$.]