1. Modules of constant Jordan type

We begin with a little background on modular representation theory to put these lectures in context. Let $G$ be a finite group and $k$ be a field. A representation of $G$ is a group homomorphism $G \to GL(n, k)$ for some $n$, or equivalently a finitely generated $kG$-module, where $kG$ is the group algebra. A representation is reducible if after a change of basis the matrices have the form \( \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \), and decomposable if after a change of basis the matrices have the form \( \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \).

- Suppose that $k$ has characteristic 0 or characteristic not dividing $|G|$. Then every reducible representation is decomposable; i.e., every invariant subspace has an invariant complement.
- In particular, it follows that every representation is a direct sum of irreducible representations (Maschke’s theorem).
- On the other hand, if $k$ has characteristic $p$ dividing $|G|$ then there exist reducible representations that are indecomposable.

Examples: An extreme case is where $G$ is a finite $p$-group in characteristic $p$. In this case there is only one irreducible representation, called the trivial module, where every group element is represented by the $1 \times 1$ matrix $(1)$. However, there are usually many indecomposable representations. If $G$ is cyclic then Jordan canonical form describes the modules. If $G$ is non-cyclic then there are infinitely many indecomposable representations.
For example, if $G = \langle g_1, g_2 \rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p$ then for each $\lambda \in k$ we have a representation of the form $g_1 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & \lambda \end{pmatrix}$, $g_2 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. These are all indecomposable and non-isomorphic.

**Representation type:** The *trichotomy theorem* (Drozd) states that for finite dimensional algebras (and in particular for the group algebras of finite groups) there are three cases:

- Finite representation type: There are finitely many isomorphism classes of indecomposable representations.
- Tame representation type: The finitely generated indecomposable representations fall into one parameter families and discrete sets in a classifiable way.
- Wild representation type: Classifying the finitely generated indecomposable modules would lead to a normal form for pairs of non-commuting matrices under simultaneous conjugation.

For finite groups, finite representation type happens if and only if the Sylow $p$-subgroups are cyclic. The remaining cases are wild, except for a few tame cases in characteristic two (dihedral, semidihedral, generalised quaternion Sylow 2-subgroups).

So how do we make progress? There are several possible approaches:

- Make general statements about modules that can be proved without obtaining a classification.
- Obtain broader categorical classification theorems.
- Restrict the type of module under consideration and study those.

There are many fruitful examples of each of these approaches. I shall concentrate on one particular class of modules, namely those of constant Jordan type. Many questions in modular representation theory reduce to the study of *elementary abelian* $p$-*groups*, i.e., groups isomorphic to $(\mathbb{Z}/p)^r$. The number $r$ is called the *rank*.

**Notation:** Let $k$ be an algebraically closed field of characteristic $p$ and let $E = \langle g_1, \ldots, g_r \rangle \cong (\mathbb{Z}/p)^r$ be an elementary abelian $p$-group. We define $X_i = g_i - 1 \in kE$, so that $X_i^p = 0$. Then we can write the group algebra $kE$ as $k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)$. If $\alpha = (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r(k)$, set

$$X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r,$$

so $X_\alpha^p = 0$. These form coset representatives for $J^2(kE)$ in $J(kE)$.

If $M$ is a finitely generated $kE$-module, the action of $X_\alpha$ on $M$ breaks up into Jordan blocks of length between 1 and $p$ with eigenvalue 0. Write $[p]^{a_p} \cdots [1]^{a_1}$ for the Jordan type.
Warning 1. If $x, y \in J(kE)$, $x - y \in J^2(kE)$, it can happen that $x$ and $y$ have different Jordan type on $M$.

Example 2. Let $p = 2$ and $r = 3$, and let $M$ be the four dimensional $kE$-module given by

$$
\begin{align*}
g_1 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \\
g_2 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \\
g_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.
\end{align*}
$$

Then $X_3$ has Jordan type $[2][1]^2$ while $X_3 + X_1X_2$ has Jordan type $[1]^4$. Note that $X_3 \equiv X_3 + X_1X_2 \pmod{J^2(kE)}$.

Definition 3. Nilpotent Jordan types are partially ordered: $X \geq Y$ if and only if for all $s > 0$ we have $\text{rank}(X^s) \geq \text{rank}(Y^s)$.

This corresponds to the dominance order on partitions. For example, $[4] > [3][1] > [2]^2 > [2][1]^2 > [1]^4$.

Definition 4. We say $x$ has maximal Jordan type on $M$ if it is maximal with respect to this partial order.

Theorem 5 (FPS 2007). (1) If $x, y \in J(kE)$ and $x - y \in J^2(kE)$ then $x$ has maximal Jordan type if and only if $y$ does.

(2) The points of $J/J^2$ of maximal Jordan type form a dense open subset.

(3) This is the same as the Jordan type at a generic point of $A^r(k)$.

So we talk of the generic Jordan type of $M$.

Definition 6 (CFP 2008). We say that a $kE$-module $M$ has constant Jordan type $[p]^a \cdots [1]^b$ if every element of $J \setminus J^2$ has this as its Jordan canonical form on $M$. The stable Jordan type is $[p - 1]^a \cdots [1]^b$.

Example 7. $E = (\mathbb{Z}/2)^4$, let $M$ be the module

$$
\begin{pmatrix}
aX_1 + bX_2 + cX_3 + dX_4 \mapsto \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d & c & b & a & 0
\end{pmatrix}.
$$


Example 8. Let $E = (\mathbb{Z}/p)^2$, $p \geq 3$, and let $M_n$ ($n \geq 2$) be the module
Then $M_n$ has constant Jordan type if and only if $n$ is divisible by $p$. The Jordan type is $[3]^{n-2}[2]^2$.

**Question 9.** Suppose that $r \geq 2$. What stable constant Jordan types can occur?

**Lemma 10** (Dade’s Lemma, 1978). If $M$ has constant Jordan type $[p]^n$ then $M$ is a free = projective = injective $kE$-module. In particular $p^{r-1}|n$.

**Theorem 11** (CFP). Every summand of a module of constant Jordan type has constant Jordan type.

**Tensor products and duals:** We make $M \otimes_k N$ into a $kE$-module via $g(m \otimes n) = gm \otimes gn$ We make $M^* = \text{Hom}_k(M, k)$ into a $kE$-module via $g(f)(m) = f(g^{-1}(m))$.

**Theorem 12** (CFP). If $M$ and $N$ have constant Jordan type then so do $M^*$ and $M \otimes_k N$.

**Warning 13.** In general $(M \otimes_k N)\downarrow_{X_a} \not\cong M\downarrow_{X_a} \otimes_k N\downarrow_{X_a}$.

**Endotrivial modules:** What modules have stable constant Jordan type [1] or $[p−1]$? If $M$ has stable constant Jordan type [1] or $[p−1]$ then $M \otimes_k M^*$ has stable constant Jordan type [1]. Then $k \rightarrow M \otimes_k M^* \rightarrow k$ has non-zero composite since $p \mid \dim M$. So $M \otimes_k M^* = k \oplus$ a module of constant Jordan type $[p]^n$. So by Dade’s lemma this is $= k \oplus$ (free). So $M$ is endotrivial. Dade (1978) classified these for $kE$, and $M \cong \Omega^n k$ ($n \in \mathbb{Z}$). These modules do indeed have stable constant Jordan type [1] if $n$ is even, $[p−1]$ if $n$ is odd.

**Single stable Jordan block:** CFP conjectured that there is no module of constant Jordan type [2] if $p \geq 5$ and $r \geq 2$. More generally we have the following.

**Theorem 14** (B, MSRI 2008). If $r \geq 2$ and $2 \leq a \leq p−2$ then there is no module of stable constant Jordan type $[a]$.

**Proof.** We have $\dim M = np + a$ and so

$$\dim \Lambda^{a+1} M = \frac{(np + a)\ldots(np)}{(a + 1) \ldots 1}.$$
It follows that $\Lambda^{a+1}M$ is free by Dade’s lemma, so $p|n$. Similarly we have
\[
\dim S^{p-a+1}M = \frac{(np + a) \ldots (np + p)}{(p - a + 1) \ldots 1}
\]
Dade’s lemma: $S^{p-a+1}M$ is also free so $p|(n+1)$, a contradiction. □

For the last line of this proof, the freeness of $S^{p-a+1}M$ we need the following lemma.

**Lemma 15** (Almkvist & Fossum 1978). As modules for $k[t]/(t^p)$, $S^i[a]$ is free provided $i < p$, $a + i > p$.

*Proof.* True if $a = p$. Downward induct on $a$ using
\[
0 \to S^i[a] \to S^{i+1}[a] \to S^{i+1}[a - 1] \to 0
\]

Here are some conjectures about modules of constant Jordan type.

**Conjecture 16** (Rickard, MSRI 2008). Suppose $r \geq 2$ and $M$ is a $kE$-module of constant Jordan type. If there are no Jordan blocks of length $i$ then the total number of Jordan blocks of length $> i$ is divisible by $p$.

This implies the previous theorem, since there are no Jordan blocks of length $a - 1$ or $a + 1$. That theorem is also implied by the following conjecture.

**Conjecture 17** (S, in CFP). Let $r \geq 2$. If $2 \leq i \leq p - 1$ and $M$ has constant Jordan type with blocks of length $i$, then it also has blocks of length either $i - 1$ or $i + 1$.

**Conjecture 18** (CFP). Let $r \geq 2$, $p \geq 5$. If there’s a module of stable constant Jordan type $[2][1]^j$ then $j \geq r - 1$.

**Definition 19** (CF, CFS). A module $M$ has the constant image property if for all $0 \neq \alpha \in \mathbb{A}^r(k)$ we have $X_{\alpha}M = \text{Rad}(M)$. Equivalently, for all $X \in J(kE) \setminus J^2(kE)$ we have $X.M = \text{Rad}(M)$.

**Lemma 20.** If $M$ has the constant image property then for all $1 \leq j \leq p$ we have $X_\alpha^j.M = \text{Rad}^p(M)$. In particular, $\text{Rad}^p(M) = 0$.

**Theorem 21.** If $M$ has the constant image property then it has constant Jordan type.

**Definition 22** (CFS). Let $E = \mathbb{Z}/p \times \mathbb{Z}/p$. The generic kernel of $M$ is
\[
\mathfrak{K}(M) = \cap_{S \subseteq P^1} \sum_{\alpha \in S} \text{Ker}(X_{\alpha}, M).
\]
Properties:

- \( \mathcal{R}(\mathcal{R}(M)) = \mathcal{R}(M) \)
- \( \mathcal{R}(M) \) has the constant image property, hence constant Jordan type.
- If \( N \subseteq M \) has the constant image property then \( N \subseteq \mathcal{R}(M) \).
- \( \ker(X_\alpha, M) \subseteq \mathcal{R}(M) \) if and only if \( X_\alpha \) has maximal rank on \( M \).
- So if \( M \) has constant Jordan type then for all \( \alpha \neq 0 \) we have \( \ker(X_\alpha, M) \subseteq \mathcal{R}(M) \).

Theorem 23. If \( M \) has constant Jordan type then \( J^{-1}\mathcal{R}(M)/J^2\mathcal{R}(M) \) also has constant Jordan type with the same number of Jordan blocks of length one.

Theorem 24 (B 2011, special case of Rickard’s conjecture). Let \( E = \mathbb{Z}/p \times \mathbb{Z}/p \) and \( M \) have constant Jordan type with no Jordan blocks of length 1. Then the total number of Jordan blocks is divisible by \( p \).

Idea of Proof. Show that \( \mathcal{R}(M)/J^2\mathcal{R}(M) \) is a sum of modules of the form \( \cdots \otimes \mathcal{R}(M) \). Then the fact that for all \( \alpha \neq 0 \) the map \( X_\alpha \) from \( J^{-1}\mathcal{R}(M)/\mathcal{R}(M) \) to \( \mathcal{R}(M)/J(\mathcal{R}(M)) \) is injective is used in order to show that the lengths of the tops of these summands are all divisible by \( p \).

Corollary 25. If \( M \) has constant Jordan type with no Jordan blocks of length \( p - 1 \) then the number of Jordan blocks of length \( p \) is divisible by \( p \).

Proof. Apply the theorem to \( \Omega(M) \).

Corollary 26. If \( M \) has constant Jordan type with no Jordan blocks of length 1 or \( p - 1 \) then the number of Jordan blocks of length between 2 and \( p - 2 \) is divisible by \( p \).

2. Vector Bundles on Projective Space

Definition 27. Let \( \mathbb{P}^{r-1} = \text{Proj} \ k[Y_1, \ldots, Y_r] \) where \( Y_1, \ldots, Y_r \) have degree one. A vector bundle on \( \mathbb{P}^{r-1} \) is a locally free sheaf of \( \mathcal{O} \)-modules, where \( \mathcal{O} \) is the structure sheaf on \( \mathbb{P}^{r-1} \).

Theorem 28 (Exercise II.5.9 of Hartshorne). There is an equivalence of categories between coherent sheaves on \( \mathbb{P}^{r-1} \) and finitely generated graded modules over \( k[Y_1, \ldots, Y_r] \) modulo finite length modules.

Twists: For a graded module \( M \) we define \( M(j)_i = M_{i+j} \). For sheaves, \( \mathcal{O}(1) \) is the twisting sheaf generated by global sections \( Y_1, \ldots, Y_r \). Then the twists of a sheaf are given by \( \mathcal{F}(j) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(1)^{\otimes j} \).
Remark 29. The only line bundles on $\mathbb{P}^{r-1}$ are $\mathcal{O}(n)$ for $n \in \mathbb{Z}$.

If $r = 2$ then every vector bundle on $\mathbb{P}^1$ is a sum of line bundles (Grothendieck).

If $r \geq 3$, it is easy to construct indecomposable vector bundles on $\mathbb{P}^{r-1}$ of every rank at least $r - 1$. Bundles of rank $r - 2$ are slightly more difficult to construct, but examples include the null correlation bundle and instanton bundles ($r$ even), some bundles of Tango (all $r$) and others.

The only known indecomposable vector bundles with rank bigger than 1 and less than $r - 2$ are:

- $\mathbb{P}^4$: the Horrocks–Mumford bundle $\mathcal{F}_{HM}$ of rank 2 with 15,000 symmetries,
- $\mathbb{P}^5$: Horrocks’ Parent bundle of rank 3,
- $\mathbb{P}^5$ in characteristic 2: the Tango bundle of rank 2,
- ... a few more of rank 2 on $\mathbb{P}^4$ and rank 3 on $\mathbb{P}^5$ in char $p$,
- ... and bundles obtained from these in obvious ways.

Vector Bundles from Modules of constant Jordan type: Let $\mathbb{P}^{r-1} = \text{Proj} k[Y_1, \ldots, Y_r]$ where $Y_i$ are functions on $\mathbb{A}^r$ defined by $Y_i(X_j) = \delta_{ij}$. Given a $kE$-module $M$, set $\tilde{M} = M \otimes_k \mathcal{O}$, a trivial bundle whose rank is equal to the dimension of $M$.

Definition 30 (FP, TAMS 2011). We define $\theta: \tilde{M}(j) \rightarrow \tilde{M}(j+1)$ via

$$\theta(m \otimes f) = \sum_i X_i m \otimes Y_i f.$$
Intuitive idea: At $\bar{\alpha} = (\lambda_1 : \cdots : \lambda_r) \in \mathbb{P}^{r-1}$ the action of $\theta$ is via
\[ m \otimes 1 \mapsto \sum_i X_i m \otimes \lambda_i = \sum_i \lambda_i X_i m \otimes 1 = X_\alpha m \otimes 1.\]
i.e., the action of $\theta$ on the copy of $M$ at $\bar{\alpha} \in \mathbb{P}^{r-1}$ is via $X_\alpha$. Notice that the twist is necessary in order to make this well defined on projective space.

**Definition 31** (BP, MSRI 2008). We define $F_i(M) = \frac{\text{Ker} \theta \cap \text{Im} \theta^{i-1}}{\text{Ker} \theta \cap \text{Im} \theta^i}$ as subquotient of $\tilde{M}$.

Now $\text{Ker} \theta$ picks out the bottoms of all Jordan blocks. $\text{Ker} \theta \cap \text{Im} \theta^i$ picks out the bottoms of the Jordan blocks of length at least $i + 1$. So $F_i$ picks out the bottoms of the Jordan blocks of length $i$. Thus $F_i(M)$ is a vector bundle iff the number of Jordan blocks of length $i$ is independent of $\bar{\alpha} \in \mathbb{P}^{r-1}$.

**Proposition 32.** $F_i(M)$ is a vector bundle for $1 \leq i \leq p$ if and only if $M$ has constant Jordan type.

More generally, define
\[ F_{i,j}(M) = \frac{\text{Ker} \theta^{j+1} \cap \text{Im} \theta^{i-j-1}}{\text{Ker} \theta^{j+1} \cap \text{Im} \theta^{i-j} + (\text{Ker} \theta^j \cap \text{Im} \theta^{i-j-1})}.\]
This captures the $(j + 1)$st layer from the bottom of the Jordan blocks of length $i$. In particular $F_{i,0}(M) = F_i(M)$. The map $\theta$ induces an isomorphism $F_{i,j}(M) \to F_{i,j-1}(M)(1)$. Therefore we have
\[ F_{i,j}(M) \cong F_i(M)(j).\]

**Observation:** $\tilde{M}$ has a filtration with filtered quotients $F_{i,j}(M)$ ($0 \leq j < i \leq p$):
Interpretation of $\theta$: Think of a homomorphism from $\tilde{M}$ to $\tilde{M}(1)$ as an $n \times n$ matrix of elements of $\text{Hom}(\mathcal{O}, \mathcal{O}(1))$. Now $\text{Hom}(\mathcal{O}, \mathcal{O}(1))$ is a vector space with basis $Y_1, \ldots, Y_r$. So we can think of $\theta$ as being a matrix of linear forms,\[ \sum Y_i \phi_M(X_i) \in \text{Mat}_n(k[Y_1, \ldots, Y_r]) \]
where $\phi_M: kE \to \text{Mat}_n(k)$ gives the representation of $E$ on $M$.

Example 33. Let $E = (\mathbb{Z}/p)^2 = \langle g_1, g_2 \rangle$, $kE = k[X_1, X_2]/(X_1^p, X_2^p)$, and let $M$ be given by $g_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $g_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

Then\[ \theta = Y_1 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + Y_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ Y_1 & 0 & 0 \\ Y_2 & 0 & 0 \end{pmatrix}. \]

The operator $\theta$ has kernel of rank two and image of rank one; $\mathcal{F}_1(M)$ and $\mathcal{F}_2(M)$ are both rank one bundles.

Example 34. More generally $M = kE/J^2(kE)$ has constant Jordan type $[2][1]^{r-1}$.

$\theta: \tilde{M} \to \tilde{M}(1)$ has $\text{Soc}(M)$ in its kernel, and its image is $\mathcal{F}_2(M)(1) \subseteq \tilde{M}(1)$. So $\mathcal{O} \cong M/\text{Soc}(M) \xrightarrow{\theta} \mathcal{F}_2(M)(1)$ gives $\mathcal{F}_2(M) \cong \mathcal{O}(-1)$. On the other hand, $\mathcal{F}_1(M)$ is $\text{Soc}(M)/\mathcal{F}_2(M)$ so\[ 0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus r} \to \mathcal{F}_1(M) \to 0. \]

The left hand map has coordinates $Y_1, \ldots, Y_r$ so this is a twisted version of the Euler sequence defining the tangent bundle\[ 0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus r} \to \mathcal{T} \to 0. \]

Thus we have\[ \mathcal{F}_1(M) \cong \mathcal{T}(-1), \quad \mathcal{F}_2(M) \cong \mathcal{O}(-1). \]

Example 35. The module $M = \text{Soc}^2(kE)$ also has constant Jordan type $[2][1]^{r-1}$. But this time $\mathcal{F}_1(M) \cong \Omega(1)$, $\mathcal{F}_2(M) \cong \mathcal{O}$ where $\Omega$ is the cotangent bundle.
Example 36 (B, MSRI 2008). If $p \geq 7$, $r = 5$, there exists a $kE$-module $M$ such that $\mathcal{F}_2(M) \cong \mathcal{F}_{HM}(-2)$. We have $\dim M = 30p^2$, and $M$ has stable constant Jordan type $[p-1][30][2][1][26]$.

Example 37. The following is an example with $E = (\mathbb{Z}/2)^6$, where $\mathcal{F}_1(M)$ is the rank two Tango bundle on $\mathbb{P}^5$.

\[
\begin{array}{cccccccc}
& & & & & f & e & d & c \\
& a & b & c & d & e & f & & \\
& a & b & c & d & e & f & & \\
& & d & f & a & c & & & \\
& e & a & a & & & & & \\
& d & b & & & & & & \\
& f & b & c & & & & & \\
& f & d & & & & & & \\
& & & & & & & & \\
\end{array}
\]

Properties of $\mathcal{F}_i$:

- $\mathcal{F}_{p-i}(\Omega M) \cong \mathcal{F}_i(M)(-p + i)$ \quad (1 \leq i \leq p - 1)
- $\mathcal{F}_i(M^*) \cong \mathcal{F}_i(M)^\vee(-i + 1)$ \quad (1 \leq i \leq p)
- $\mathcal{F}_1(M \otimes_k N) \cong \bigoplus_{i=1}^{p-1} \mathcal{F}_i(M) \otimes \mathcal{F}_i(N)(i - 1)$

The sequence $0 \to \Omega M \to P_M \to M \to 0$ induces

$0 \to \mathcal{F}_p(\Omega M) \to \mathcal{F}_p(P_M) \to \mathcal{F}_p(M) \to 0$.

This is not exact, but has homology only in the middle, where it is

$\bigoplus_{i=1}^{p-1} \mathcal{F}_i(M)(-p + i)$. 
**Theorem 38** (Realisation Theorem (BP, MSRI 2008)). Given a vector bundle $\mathcal{F}$ of rank $s$ on $\mathbb{P}^{r-1}$, there exists a $kE$-module $M$ of stable constant Jordan type $[1]^s$ such that

- if $p = 2$ then $\mathcal{F}_1(M) \cong \mathcal{F}$
- if $p$ is odd then $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$

where $F: \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the Frobenius map.

Let us outline the proof of the realisation theorem. We begin with $p = 2$. Given $\mathcal{F}$, Hilbert’s syzygy theorem gives a resolution

$$0 \to \sum_{j=1}^{m_r} \mathcal{O}(a_{r,j}) \to \cdots \to \sum_{j=1}^{m_1} \mathcal{O}(a_{1,j}) \to \sum_{j=1}^{m_0} \mathcal{O}(a_{0,j}) \to \mathcal{F} \to 0.$$ 

If $a > b$ then $\text{Hom}(\mathcal{O}(a), \mathcal{O}(b)) = 0$, while if $a \leq b$ it is the space of degree $b - a$ polynomials in $Y_1, \ldots, Y_r$. Now mimic this with representations of $kE$. We have

$$H^*(E, k) = k[y_1, \ldots, y_r].$$

We have $\mathcal{F}_1(\Omega^{-a}(k)) \cong \mathcal{O}(a)$ and provided $a \leq b$

$$\text{Hom}_{kE}(\Omega^{-a}(k), \Omega^{-b}(k)) \cong H^{b-a}(E, k)$$

is the space of degree $b - a$ polynomials in $y_1, \ldots, y_r$.

**Lemma 39.** Representatives $\hat{y}_i: \Omega^{n+1}(k) \to \Omega^n(k)$ of $y_i \in H^1(E, k)$ can be found so that $\hat{y}_i\hat{y}_j = \hat{y}_j\hat{y}_i: \Omega^{n+2}(k) \to \Omega^n(k)$.

Define a $k$-algebra homomorphism

$$\rho: H^*(E, k) = k[y_1, \ldots, y_r] \to k[Y_1, \ldots, Y_r]$$

by $\rho(y_i) = Y_i$. Then representing an element $\zeta \in H^*(E, k)$ by a cocycle $\hat{\zeta}: \Omega^{n+1}(k) \to \Omega^n(k)$ the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{O}(-n-j) & \xrightarrow{\rho(\zeta)} & \mathcal{O}(-j) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{F}_1(\Omega^{n+1}(k)) & \xrightarrow{\mathcal{F}_1(\hat{\zeta})} & \mathcal{F}_1(\Omega^n(k))
\end{array}
$$

Now take a resolution of $\mathcal{F}$

$$0 \to \sum_{j=1}^{m_r} \mathcal{O}(a_{r,j}) \to \cdots \to \sum_{j=1}^{m_1} \mathcal{O}(a_{1,j}) \to \sum_{j=1}^{m_0} \mathcal{O}(a_{0,j}) \to \mathcal{F} \to 0.$$ 

Apply $\rho^{-1}$ to the entries in the maps in this complex to get

$$0 \to \sum_{j=1}^{m_r} \Omega^{-a_{r,j}}(k) \to \cdots \to \sum_{j=1}^{m_1} \Omega^{-a_{1,j}}(k) \to \sum_{j=1}^{m_0} \Omega^{-a_{0,j}}(k) \to 0.$$
Rickard has a “totalisation” functor $D^b(kE) \to \text{stmod}(kE)$. Applying this to the above complex gives a module $M$ with $\mathcal{F}_1(M) \cong \mathcal{F}$.

When $p$ is odd, the best we can do is to get a module of type $[p]a[1]^b$ with $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$. We’ll see using Chern classes why this is best possible. We have

$$H^i(E, k) \cong \Lambda(y_1, \ldots, y_r) \otimes k[x_1, \ldots, x_r]$$

with $\deg(y_i) = 1$, $\deg(x_i) = 2$.

**Lemma 40.** Representatives $\hat{x}_i: \Omega^{n+2}(k) \to \Omega^n(k)$ of $x_i \in H^2(E, k)$ can be found so that $\hat{x}_i \hat{x}_j = \hat{x}_j \hat{x}_i: \Omega^{n+4}(k) \to \Omega^n(k)$.

We have $\mathcal{F}_1(\Omega^{-2a}(k)) \cong \mathcal{O}(pa)$. Define a $k$-algebra homomorphism $\rho: k[x_1, \ldots, x_r] \to k[Y_1, \ldots, Y_r]$ by $\rho(x_i) = Y_i^p$. Then representing an element $\zeta \in k[x_1, \ldots, x_r]$ by $\hat{\zeta}: \Omega^j(k)$ the following diagram commutes:

$$\mathcal{O}(-p(n+j)) \xrightarrow{\rho(\zeta)} \mathcal{O}(-pj) \xrightarrow{\cong} \mathcal{F}_1(\Omega^{2(n+j)}(k)) \xrightarrow{\mathcal{F}_1(\hat{\zeta})} \mathcal{F}_1(\Omega^{2j}(k)) \cong \mathcal{F}.$$  

Now take a resolution of $\mathcal{F}$

$$0 \to \sum_{j=1}^{m_r} \mathcal{O}(a_{r,j}) \to \cdots \to \sum_{j=1}^{m_1} \mathcal{O}(a_{1,j}) \to \sum_{j=1}^{m_0} \mathcal{O}(a_{0,j}) \to \mathcal{F} \to 0.$$  

Each map is a matrix of polynomials in $Y_1, \ldots, Y_r$. Replace each $Y_i$ by $Y_i^p$ to get a complex

$$0 \to \sum_{j=1}^{m_r} \mathcal{O}(pa_{r,j}) \to \cdots \to \sum_{j=1}^{m_1} \mathcal{O}(pa_{1,j}) \to \sum_{j=1}^{m_0} \mathcal{O}(pa_{0,j}) \to F^*(\mathcal{F}) \to 0.$$  

This is a resolution of $F^*(\mathcal{F})$, where $F: \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the Frobenius map induced by

$$k[Y_1, \ldots, Y_r] \to k[Y_1, \ldots, Y_r]$$  

$$Y_i \mapsto Y_i^p$$

The entries in the maps are now in the image of $\rho$. Apply $\rho^{-1}$ to the entries to get a complex

$$0 \to \sum_{j=1}^{m_r} \Omega^{-2a_{r,j}}(k) \to \cdots \to \sum_{j=1}^{m_1} \Omega^{-2a_{1,j}}(k) \to \sum_{j=1}^{m_0} \Omega^{-2a_{0,j}}(k) \to 0.$$
Again apply Rickard’s totalisation functor $\mathcal{D}^b(kE) \to \text{stmod}(kE)$ to the above complex to get a module $M$ with $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$.

3. Chern Classes

The Chow group $A^*(\mathbb{P}^{r-1})$ is isomorphic to $\mathbb{Z}[h]/(h^r)$. Given a vector bundle $\mathcal{F}$ on $\mathbb{P}^{r-1}$, there is a Chern polynomial

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F})h + \cdots + c_{r-1}(\mathcal{F})h^{r-1} \in A^*(\mathbb{P}^{r-1})$$

whose coefficients $c_i(\mathcal{F})$ are the Chern numbers of $\mathcal{F}$. We’ll construct the Chern polynomial in this lecture without reference to the general definition of Chow group. Recall

**Theorem 41** (Exercise II.5.9 of Hartshorne). There is an equivalence of categories between coherent sheaves on $\mathbb{P}^{r-1} = \text{Proj } k[Y_1, \ldots, Y_r]$ and finitely generated graded modules over $k[Y_1, \ldots, Y_r]$ modulo finite length modules.

**Definition 42.** If $M = \bigoplus_{j \in \mathbb{Z}} M_j$ is a finitely generated graded module over $R = k[Y_1, \ldots, Y_r]$ then the Poincaré series (or Hilbert series) of $M$ is

$$p_M(t) = \sum_{j \in \mathbb{Z}} t^j \dim_k M_j.$$

**Lemma 43** (Hilbert, Serre). The Poincaré series of a finitely generated $R$-module takes the form $p_M(t) = \frac{f(t)}{(1 - t)^r}$ where $f(t)$ is a Laurent polynomial.

If $M$ is a finite length module then $p_M(t)$ is a Laurent polynomial. i.e., $f(t)$ is divisible by $(1 - t)^r$.

**Definition 44.** We define the rank of $M$ to be the positive integer $f(1)$. This is equal to the dimension of the ungraded vector space

$$k(Y_1, \ldots, Y_r) \otimes_{k[Y_1, \ldots, Y_r]} M$$

over the field $k(Y_1, \ldots, Y_r)$.

**Lemma 45.** If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of graded $R$-modules then $p_{M_2}(t) = p_{M_1}(t) + p_{M_3}(t)$.

If $M$ corresponds to a vector bundle $\mathcal{F}$ on $\mathbb{P}^{r-1}$ then the rank is the dimension of the vector space at each point.

**Definition 46.** We define the Chow ring of $R$ to be the truncated polynomial ring $A^*(R) = \mathbb{Z}[h]/(h^r)$. 

Definition 47. If \( p_M(t) = \frac{\sum_j a_j t^j}{(1 - t)^r} \) then we define the Chern polynomial of \( M \) to be

\[
c(M) = \prod_j (1 + jh)^{a_j} \in A^*(R) = \mathbb{Z}[h]/(h^r).
\]

Some \( a_j \) may be negative, but \((1 + jh)\) is invertible in \( A^*(R) \).

The Chern numbers of \( M \) are the coefficients

\[
c(M) = 1 + c_1(M)h + \cdots + c_{r-1}(M)h^{r-1}
\]

and by convention \( c_0(M) = 1 \).

The Chern character of \( M \) is defined to be

\[
\text{Ch}(M) = \sum_j a_je^{jh} \in A^*_R = \mathbb{Q} \otimes \mathbb{Z} A^*(R) = \mathbb{Q}[h]/(h^r).
\]

\[
\text{Ch}(M) = \text{rank} (M) + c_1 h + \frac{1}{2}(c_1^2 - 2c_2)h^2 + \cdots
\]

Lemma 48. If \( 0 \to M_1 \to M_2 \to M_3 \to 0 \) is a short exact sequence then

(i) \( c(M_2) = c(M_1)c(M_3) \) — i.e., \( c_j(M_2) = \sum_{i=0}^j c_i(M_1)c_{j-i}(M_3) \)

(ii) \( \text{Ch}(M_2) = \text{Ch}(M_1) + \text{Ch}(M_3) \).

Lemma 49. If \( M \) and \( M' \) are equivalent modulo finite length modules then \( c(M) = c(M') \) and \( \text{Ch}(M) = \text{Ch}(M') \).

Proof. For \( \text{Ch}(M) \), easy: \( \text{Ch}(k[n]) = \sum_j (-1)^j \binom{r}{j} e^{(j+n)h} = e^{nh}(1 - e^h)^r \)

and \( 1 - e^h \) is divisible by \( h \).

For \( c(M) \), need \( c(k[n]) = 1 \). This follows from:

\[
c(k[n]) = \prod_{j=0}^r (1 + (j+n)h)^{(-1)^j \binom{r}{j}} \equiv 1 \pmod{h^r}.
\]

Definition 50. If a coherent sheaf \( \mathcal{F} \) on \( \mathbb{P}^{r-1} \) corresponds to a finitely generated graded \( k[Y_1, \ldots, Y_r] \)-module \( M \) then we define \( c(\mathcal{F}) = c(M) \) and \( \text{Ch}(\mathcal{F}) = \text{Ch}(M) \).

Exercise: Show that \( c(\mathcal{F}(1), h) = (1 + h)^{\text{rank} \mathcal{F}} c(\mathcal{F}, \frac{h}{1+h}) \).

Fact: For a vector bundle \( c_i(\mathcal{F}) = 0 \) for \( i > \text{rank} \mathcal{F} \).

Next we discuss congruences on Chern classes.

Lemma 51. For a vector bundle \( \mathcal{F} \) of rank \( s \) in \( \mathbb{Z}[h]/(h^r) \) we have

\[
c(\mathcal{F})c(\mathcal{F}(1)) \cdots c(\mathcal{F}(p-1)) \equiv 1 - sh^{p-1} \pmod{p, h^{2p-2}}.
\]
Proof. Recall that if \( p_M(t) = \sum_j a_j t^j / (1-t)^r \) then \( c(F) = \prod_j (1+jh)^{a_j} \).

\[
c(F) \ldots c(F(p-1)) = \prod_j ((1+jh) \ldots (1+(j+p-1)h))^{a_j}.
\]

Since \( x(x+y)(x+2y) \ldots (x+(p-1)y) \equiv x^p - xy^{p-1} \pmod{p} \), this

\[
eq \prod_j ((1+jh)^p - (1+jh)h^{p-1})^{a_j} \quad \pmod{p}
\]

\[
eq \prod_j (1 - h^{p-1} + (j^p - j)h)^{a_j} \quad \pmod{p}
\]

\[
\equiv 1 - \sum_j a_j h^{p-1} \quad \pmod{p, h^{2p-2}}.
\]

\[\square\]

\textbf{Theorem 52 (BP).} Suppose \( r \geq 2 \), and let \( M \) be a \( kE \)-module of stable constant Jordan type \([1]^s\). Then \( p|c_i(F_1(M)) \) for \( 1 \leq i \leq p-2 \).

\textit{Proof.} \( \tilde{M} \) has a filtration with filtered quotients

\[
\mathcal{F}_1(M), \mathcal{F}_p(M), \mathcal{F}_p(M)(1), \ldots, \mathcal{F}_p(M)(p-1).
\]

Therefore \( c(F_1(M))c(F_p(M))c(F_p(M)(1)) \ldots c(F_p(M)(p-1)) = 1 \). By the lemma, it follows that \( c(F_1(M)) \equiv 1 + sh^{p-1} \pmod{p, h^{2p-2}}. \)

\[\square\]

If \( p = 2 \) this gives no information, but for \( p \) odd it gives a genuine restriction on the vector bundles that can occur this way. In particular, it throws light on the realisation theorem.

\textbf{Remark 53.} If \( F \) is the Frobenius map then \( c(F^*(\mathcal{F}), h) = c(\mathcal{F}, ph) \). So the condition is satisfied by \( F^*(\mathcal{F}) \).

\textbf{Example 54.} The rank two Horrocks–Mumford bundle \( \mathcal{F}_{HM} \) on \( \mathbb{P}^4 \) has \( c_1(\mathcal{F}_{HM}(i)) = 2i + 5 \) and \( c_2(\mathcal{F}_{HM}(i)) = i^2 + 5i + 10 \). So no twist of \( \mathcal{F}_{HM} \) can occur as \( \mathcal{F}_1(M) \) for a module of stable constant Jordan type \([1]^2\). But by the realisation theorem there is a module \( M \) of stable constant Jordan type \([1]^2\) with \( \mathcal{F}_1(M) \cong F^*(\mathcal{F}_{HM}) \).

A similar analysis of Chern classes shows that for large rank and large primes, the only small stable constant Jordan type is \([1]^t\):

\textbf{Theorem 55 (B, 2010).} If a module has stable constant Jordan type \([a_1][a_2] \ldots [a_t] \) with \( a = \sum a_i \leq \min(r-1, p-2) \) then \( a_1 = \cdots = a_t = 1 \).
Proof. We have
\[ p^{-2} \prod_{j=1}^{p-2} c(\mathcal{F}_j(M))c(\mathcal{F}_j(M)(1)) \equiv 1 \pmod{(p, h^{p-1})}. \]
This is a polynomial of degree \( a \leq p - 2 \). Also \( a \leq r - 1 \) so this can be read as an equality in \( \mathbb{F}_p[h] \). The only units in this ring are the constants. So for \( j \geq 2 \) both \( c(\mathcal{F}_j(M)) \) and \( c(\mathcal{F}_j(M)(1)) \) have to be 1 mod \( p \). But \( c_1(\mathcal{F}_j(M)(1)) = c_1(\mathcal{F}_j(M)) + \text{ rank } \mathcal{F}_j(M) \) so \( p | \text{ rank } \mathcal{F}_j(M) \), a contradiction. \( \square \)

This proves a weak form of the conjecture of CFP. Recall:

**Conjecture 56.** Let \( r \geq 2, \ p \geq 5 \). If \( M \) has stable constant Jordan type \([2][1]\) then \( j \geq r - 1 \).

**Corollary 57.** If \( M \) has stable constant Jordan type \([2][1]\) and \( p \geq j + 4 \) then \( j \geq r - 2 \).

The smallest case where there’s a discrepancy between the conjecture and the corollary is type \([2][1]\) for \( r = 3, \ p \geq 5 \). In this case it can be proved that \( p \equiv 1 \pmod{3} \).

**Chern roots:** The Chern polynomial \( c(\mathcal{F}) \in \mathbb{Z}[h]/(h^r) \) has a unique lift to \( \mathbb{Z}[h] \) of degree \( \leq r - 1 \), also denoted \( c(\mathcal{F}) \). Factorise it in \( \mathbb{C}[h] \):
\[ c(\mathcal{F}) = \prod_j (1 + \alpha_j h). \]
The algebraic integers \( \alpha_j \) are the Chern roots of \( \mathcal{F} \).
\[ c_1 = \sum_i \alpha_i \quad c_2 = \sum_{i<j} \alpha_i \alpha_j \quad \ldots \]

The number of Chern roots for a coherent sheaf is not well defined, but for a vector bundle it can be taken as the rank.

**Definition 58 (Power sums).** \( s(\mathcal{F}, h) \in \mathbb{Z}[h]/(h^r) \) is defined by
\[ -s(\mathcal{F}, -h) = \frac{hc(\mathcal{F}, h)}{c(\mathcal{F}, h)} \quad s(\mathcal{F}, h) = s_1(\mathcal{F})h + s_2(\mathcal{F})h^2 + \ldots \]
We have \( s_1 = c_1, \ s_2 = c_1^2 - 2c_2, \ldots \)

**Calculation:** \( s_n(\mathcal{F}) = \sum_j a_j j^n = \sum_j \alpha_j^n \).

**Theorem 59.** If \( 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \) then \( s_n(\mathcal{F}_2) = s_n(\mathcal{F}_1) + s_n(\mathcal{F}_3) \).

**Proof.** This follows from \( s_n(\mathcal{F}) = \sum_j a_j j^n \). \( \square \)

**Theorem 60.** If \( f(x) \) is any polynomial of degree at most \( r - 1 \) with \( f(0) = 0 \) then
\[ \sum_j f(\alpha_j) = \sum_j a_j f(j). \]
Proof. True for \( f(x) = x^n \) for \( 1 \leq n \leq r - 1 \) by previous frame. \( \square \)

**Consequence (Schwarzenberger’s conditions):** If \( n \in \mathbb{Z} \Rightarrow f(n) \in \mathbb{Z} \) then \( \sum_j f(\alpha_j) \in \mathbb{Z} \).

Examples of such polynomials are binomials \( f(n) = \binom{n}{i} \).

For example, on \( \mathbb{P}^3 \) we have \( c_1c_2 + c_3 \equiv 0 \mod 2 \).

For a rank two bundle on \( \mathbb{P}^4 \) we have \( c_2(c_2+1-3c_1-2c_1^2) \equiv 0 \mod 12 \).

**Theorem 61.** \( \text{Ch}(\mathcal{F}) = \text{rank } \mathcal{F} + \sum_j (e^{\alpha_j h} - 1) \).

**Remark 62.** If we assume that the number of Chern roots is the rank of \( \mathcal{F} \) this reads as \( \text{Ch}(\mathcal{F}) = \sum_j e^{\alpha_j h} \).

Proof. We have
\[
\text{Ch}(\mathcal{F}) = \sum_j a_j e^{jh} = \text{rank } \mathcal{F} + \sum_j a_j (e^{jh} - 1)
= \text{rank } \mathcal{F} + \sum_j \sum_{n=1}^{r-1} \frac{a_j n! h^n}{n!}.
\]

Apply Theorem 60:
\[
\text{Ch}(\mathcal{F}) = \text{rank } \mathcal{F} + \sum_j \sum_{n=1}^{r-1} \frac{\alpha_j n! h^n}{n!} = \text{rank } \mathcal{F} + \sum_j (e^{\alpha_j h} - 1). \quad \square
\]

**Cohomology of sheaves:** The global section functor \( \mathcal{F} \mapsto \Gamma(\mathcal{F}) = \Gamma(\mathbb{P}^{r-1}, \mathcal{F}) \) is left exact but not right exact. So it has right derived functors \( H^i(\mathcal{F}) = H^i(\mathbb{P}^{r-1}, \mathcal{F}) \). e.g. \( H^0(\mathcal{F}) = \Gamma(\mathcal{F}) \). These vanish for \( i \geq r \). A short exact sequence \( 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \) gives
\[
0 \to H^0(\mathcal{F}_1) \to H^0(\mathcal{F}_2) \to H^0(\mathcal{F}_3) \to H^1(\mathcal{F}_0) \to \cdots \to H^{r-1}(\mathcal{F}_3) \to 0.
\]

**Definition 63.** The *Euler characteristic* of \( \mathcal{F} \) is
\[
\chi(\mathcal{F}) = \sum_{i=0}^{r-1} (-1)^i \dim H^i(\mathcal{F}).
\]

**Lemma 64.** If \( 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \) then \( \chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3) \).

**Theorem 65** (Schwarzenberger). If \( \mathcal{F} \) is a coherent sheaf on \( \mathbb{P}^{r-1} \) then
\[
\chi(\mathcal{F}) = \text{rank } \mathcal{F} + \sum_j \left( \binom{\alpha_j + r - 1}{r - 1} - 1 \right).
\]
Proof. Both sides are additive over short exact sequences. Every coherent sheaf has a finite resolution by sums of line bundles. So it suffices to prove the theorem for \( F = \mathcal{O}(j) \). Serre calculated \( H^i(\mathcal{O}(j)) \): its dimension is \( \binom{r-j}{r-1} \) if \( i = 0 \); \( j \geq 0 \), \( \binom{r-j}{r-j} \) if \( i = r-1 \), \( j \leq -r \), zero otherwise. So it is true by direct calculation for \( \mathcal{O}(j) \). □

**Theorem 66** (Hirzebruch-Riemann-Roch). *The Euler characteristic \( \chi(\mathcal{F}) \) is the coefficient of \( h^{r-1} \) in \( \left( \frac{h}{1-e^{-h}} \right)^r \text{Ch}(\mathcal{F}) \).*

**Remark 67.** The expression \( \left( \frac{h}{1-e^{-h}} \right)^r \) is the Todd class of the tangent bundle of \( \mathbb{P}^{r-1} \).

**Remark 68.** For simplicity let’s assume that the number of Chern roots is rank \( \mathcal{F} \) so that \( \text{Ch}(\mathcal{F}) = \sum_j e^{\alpha_j h} \).

Proof. Cauchy’s integral formula: coefficient of \( h^{r-1} \) is

\[
\frac{1}{2\pi i} \oint \left( \frac{h}{1-e^{-h}} \right)^r \frac{\text{Ch}(\mathcal{F})}{h^r} \, dh = \sum_j \frac{1}{2\pi i} \oint \frac{e^{\alpha_j h}}{(1-e^{-h})^r} \, dh
\]

Substitute \( z = 1 - e^{-h} \), \( dh = dz/(1-z) \), \( e^{\alpha_j h} = 1/(1-z)^{\alpha_j} \)

\[
= \sum_j \frac{1}{2\pi i} \oint \frac{dz}{z^r (1-z)^{\alpha_j+1}}
\]

\[
= \sum_j \frac{1}{2\pi i} \oint z^{-r} (1 + (\alpha_j + 1)z + \frac{\alpha_j + 2}{2} z^2 + \ldots) dz
\]

\[
= \sum_j \left( \frac{\alpha_j + r - 1}{r - 1} \right) = \chi(\mathcal{F}). \quad \square
\]

The following theorem is a typical example of an application of the Hirzebruch–Riemann–Roch theorem to modules of constant Jordan type for \( p = 2 \).

**Theorem 69** (B, 2010). *Let \( k \) have char 2. If \( M \) has constant Jordan type \( [2]^n[1]^m \) with \( m \leq r-3 \) then one of the following occurs:

(i) \( n \) is congruent to one of \( 0, -1, \ldots, -m \) modulo \( 2^{r-1} \), or

(ii) \( r \leq 6 \), or

(iii) there is a new vector bundle of low rank on projective space of dimension at least six.*

Let’s see why this is. Suppose \( M \) has constant Jordan type \( [2]^n[1]^m \).

The sheaf \( \tilde{M} \) has a filtration with quotients \( \mathcal{F}_2(M), \mathcal{F}_2(M)(1) \) and
So if $\mathcal{F}_1(M)$ is a sum of line bundles $\mathcal{O}(a_1), \ldots, \mathcal{O}(a_m)$ then

$$(1 + e^h)\text{Ch}(\mathcal{F}_2(M)) + e^{a_1h} + \cdots + e^{a_mh} = \text{Ch}(\widetilde{M}) = 2n + m.$$ 

So by the Hirzebruch–Riemann–Roch theorem $\chi(\mathcal{F}_2(M))$ is the coefficient of $h^{r-1}$ in

$$\left(\frac{h}{1 - e^{-h}}\right)^r \left(\frac{2n + m - e^{a_1h} - \cdots - e^{a_mh}}{1 + e^h}\right).$$

**Calculation:** This differs by an integer from

$$\frac{2n + m - \sum_{i=1}^m (-1)^{a_i}}{2^r}.$$ 

So $n$ is congruent mod $2^{r-1}$ to minus the number of odd $a_i$.

**Example 70.** The Tango example of type $[2]^{14}[1]^2$ and rank 6 shows why we need (ii).

**Possible Jordan types for $(\mathbb{Z}/2)^3$:** $[2]^n[1]^m$

![Diagram](image1.png)

**Possible Jordan types for $(\mathbb{Z}/2)^4$:** $[2]^n[1]^m$

![Diagram](image2.png)

**Nilvarieties of constant Jordan type**

**Definition 71.** A *nilvariety* of constant Jordan type $t$ and rank $r$ consists of nilpotent matrices $A_1, \ldots, A_r$ such that for all $0 \neq \alpha = (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r(k)$ the Jordan canonical form of $\lambda_1 A_1 + \cdots + \lambda_r A_r$ is $t$.

**Example 72.** The matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$
span a nilvariety, but do not commute. For all values of $Y_1$ and $Y_2$, not both zero, the matrix $Y_1A_1 + Y_2A_2$ has a single Jordan block with eigenvalue zero.

**Theorem 73.** Let $p \geq 3$ and let $M$ be a rank $r$ nilvariety of constant Jordan type $[p]^n$. Then

$$p\lfloor \frac{r-1}{p-1}\rfloor n$$

where $\lfloor \frac{r-1}{p-1}\rfloor$ denotes the largest integer less than or equal to $\frac{r-1}{p-1}$.

**Proof.** Hirzebruch–Riemann–Roch. □

This is sharp for $p = 3$ in characteristic 3, because of tensor products of copies of the previous example.

What happens for $p \geq 5$?

**Nilvarieties with a single Jordan block**

**Theorem 74** (Causa, Re and Teodorescu). Let $M$ be a nilvariety of rank $r$ and constant Jordan type $[m]$. Then $r \leq 2$, and if $r = 2$ then $m$ is odd.

**Proof.** Suppose that $M$ is a nilvariety of constant Jordan type $[m]$. Then $F_m(M)$ is a line bundle, so we have $F_m(M) \cong O(a)$ for some integer $a$. The bundle $\tilde{M} \cong O^{\oplus m}$ has a filtration with filtered quotients $O(a), O(a+1), \ldots, O(a+m-1)$ and so

$$0 = c_1(\tilde{M}) = a + (a+1) + \cdots + (a+m-1) = ma + m(m-1)/2.$$

Thus $a = -(m-1)/2$ and so $m$ is odd. If $r \geq 3$ then for all $i$ and $j$ we have $\text{Ext}^1(O(i), O(j)) \cong H^1(\mathbb{P}^{r-1}, O(j-i)) = 0$ so the filtration splits, giving $O^{\oplus m} \cong O(a) \oplus O(a+1) \oplus \cdots \oplus O(a+m-1)$, which contradicts the Krull–Schmidt Theorem for vector bundles. □
Summary Sheet

- $k = \overline{k}$ is a field of characteristic $p$.
- $E = \langle g_1, \ldots, g_r \rangle \cong (\mathbb{Z}/p)^r$
- $X_i = g_i - 1 \in kE$, $X_i^p = 0$.
- $kE = k[X_1, \ldots, X_r]/(X_i^p, \ldots, X_r^p)$
- If $\alpha = (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r(k)$ then $X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r \in kE$.
- Generic kernel: $\mathcal{K}(M) = \bigcap_{S \subseteq \mathbb{P}^1} \bigcap_{\text{cofinite} \Delta \in S} \text{Ker}(X_\alpha, M)$
- $\mathbb{P}^{r-1} = \text{Proj} k[Y_1, \ldots, Y_r]$, $Y_i(X_j) = \delta_{ij}$
- $\tilde{M} = M \otimes_k \mathcal{O}$
- $\theta: \tilde{M}(j) \to \tilde{M}(j+1)$
- $\theta(m \otimes f) = \sum_i X_i m \otimes Y_i f$
- $\mathcal{F}_i(M) = \frac{\text{Ker}\theta \cap \text{Im}\theta^{i-1}}{\text{Ker}\theta \cap \text{Im}\theta^{i}}$ as subquotient of $\tilde{M}$.
- $\mathcal{F}_{i,j}(M) = \frac{(\text{Im}\theta^{j+1} \cap \text{Im}\theta^{i-j}) + (\text{Im}\theta^j \cap \text{Im}\theta^{i-j-1})}{\text{Im}\theta^{j+1} \cap \text{Im}\theta^{i-j}}$.
- $\mathcal{F}_{i,j}(M) \cong \mathcal{F}_{i}(M)(j)$.
- Euler sequence: $0 \to 0 \to O(1)^{\oplus r} \to \mathcal{F} \to 0$.
- Chow group $A^r(\mathbb{P}^{r-1}) \cong \mathbb{Z}[h]/(h^r)$.
- Chern polynomial: if $p_M(t) = \sum_j a_j t^j \in (1-t)^r$ then
  \[ c(M) = 1 + c_1(M) h + \cdots + c_{r-1}(M) h^{r-1} = \prod_j (1 + jh)^{a_j} \in \mathbb{Z}[h]/(h^r). \]
- Chern character: $\text{Ch}(M) = \sum_j a_j e^{jh} \in \mathbb{Q}[h]/(h^r)$.
- Twists: $c(\mathcal{F}(1), h) = (1 + h)^{\text{rank} \mathcal{F}} c(\mathcal{F}, \frac{h}{1+h})$.
- $c(\mathcal{F}) c(\mathcal{F}(p-1)) \equiv 1 - (\text{rank} \mathcal{F}) h^{p-1} \pmod {p, h^{2p-2}}$.
- Power sums: $-s(\mathcal{F}, -h) = \frac{h}{c(\mathcal{F}, h)} c(\mathcal{F}, h)$
- $s_n(\mathcal{F}) = \sum_j a_j j^n = \sum_j a_j^n$ for $1 \leq n < r$.
- Schwarzeneberger: $\sum_j \binom{\alpha_j + s}{m} \in \mathbb{Z}$.
- Euler characteristic: $\chi(\mathcal{F}) = \sum_i(-1)^i \dim H^i(\mathbb{P}^{r-1}, \mathcal{F})$.
- Hirzebruch–Riemann–Roch: $\chi(\mathcal{F}) = \sum_j \binom{\alpha_j + r - 1}{r - 1}$
  is the coefficient of $h^{r-1}$ in $\left(\frac{h}{1-e^{-h}}\right)^r \text{Ch}(\mathcal{F})$. 