# MODULES OF CONSTANT JORDAN TYPE AND VECTOR BUNDLES ON PROJECTIVE SPACE

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DAVE BENSON

Dramatis Personæ

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| B | ... |
| C | Jon Carlson |
| F | Eric Friedlander |
| P | Julia Pevtsova |
| S | Andrei Suslin |

## 1. Modules of constant Jordan type

We begin with a little background on modular representation theory to put these lectures in context. Let $G$ be a finite group and $k$ be a field. A *representation* of $G$ is a group homomorphism $G \to GL(n,k)$ for some $n$, or equivalently a finitely generated $kG$-module, where $kG$ is the *group algebra*. A representation is *reducible* if after a change of basis the matrices have the form $\left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)$, and *decomposable* if after a change of basis the matrices have the form $\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$.

- Suppose that $k$ has characteristic 0 or characteristic not dividing $|G|$. Then every reducible representation is decomposable; i.e., every invariant subspace has an invariant complement.
- In particular, it follows that every representation is a direct sum of irreducible representations (Maschke's theorem).
- On the other hand, if $k$ has characteristic $p$ dividing $|G|$ then there exist reducible representations that are indecomposable.

**Examples:** An extreme case is where $G$ is a finite $p$-group in characteristic $p$. In this case there is only one irreducible representation, called the *trivial module*, where every group element is represented by the $1 \times 1$ matrix $(1)$. However, there are usually many indecomposable representations. If $G$ is cyclic then Jordan canonical form describes the modules. If $G$ is non-cyclic then there are *infinitely many* indecomposable representations.

For example, if $G = \langle g_1, g_2 \rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p$ then for each $\lambda \in k$ we have a representation of the form $g_1 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $g_2 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)$. These are all indecomposable and non-isomorphic.

**Representation type:** The *trichotomy theorem* (Drozd) states that for finite dimensional algebras (and in particular for the group algebras of finite groups) there are three cases:

- Finite representation type: There are finitely many isomorphism classes of indecomposable representations.
- Tame representation type: The finitely generated indecomposable representations fall into one parameter families and discrete sets in a classifiable way.
- Wild representation type: Classifying the finitely generated indecomposable modules would lead to a normal form for pairs of non-commuting matrices under simultaneous conjugation.

For finite groups, finite representation type happens if and only if the Sylow $p$-subgroups are cyclic. The remaining cases are wild, except for a few tame cases in characteristic two (dihedral, semidihedral, generalised quaternion Sylow 2-subgroups).

So how do we make progress? There are several possible approaches:

- Make general statements about modules that can be proved without obtaining a classification.
- Obtain broader categorical classification theorems.
- Restrict the type of module under consideration and study those.

There are many fruitful examples of each of these approaches. I shall concentrate on one particular class of modules, namely those of *constant Jordan type*. Many questions in modular representation theory reduce to the study of *elementary abelian $p$-groups*, i.e., groups isomorphic to $(\mathbb{Z}/p)^r$. The number $r$ is called the *rank*.

**Notation:** Let $k$ be an algebraically closed field of characteristic $p$ and let $E = \langle g_1, \ldots, g_r \rangle \cong (\mathbb{Z}/p)^r$ be an elementary abelian $p$-group. We define $X_i = g_i - 1 \in kE$, so that $X_i^p = 0$. Then we can write the group algebra $kE$ as $k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)$. If $\alpha = (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r(k)$, set

$$X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r,$$

so $X_\alpha^p = 0$. These form coset representatives for $J^2(kE)$ in $J(kE)$.

If $M$ is a finitely generated $kE$-module, the action of $X_\alpha$ on $M$ breaks up into Jordan blocks of length between 1 and $p$ with eigenvalue 0. Write $[p]^{a_p} \ldots [1]^{a_1}$ for the Jordan type.

**Warning 1.** If $x,\ y \in J(kE)$, $x - y \in J^2(kE)$, it can happen that $x$ and $y$ have different Jordan type on $M$.

**Example 2.** Let $p = 2$ and $r = 3$, and let $M$ be the four dimensional $kE$-module given by

$$g_1 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad g_2 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad g_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then $X_3$ has Jordan type $[2]\,[1]^2$ while $X_3 + X_1X_2$ has Jordan type $[1]^4$. Note that $X_3 \equiv X_3 + X_1X_2 \pmod{J^2(kE)}$.

**Definition 3.** Nilpotent Jordan types are partially ordered: $X \geq Y$ if and only if for all $s > 0$ we have $\mathrm{rank}(X^s) \geq \mathrm{rank}(Y^s)$.

This corresponds to the *dominance order* on partitions. For example,

$$[4] > [3][1] > [2]^2 > [2][1]^2 > [1]^4.$$

**Definition 4.** We say $x$ has *maximal* Jordan type on $M$ if it is maximal with respect to this partial order.

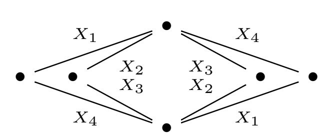
**Theorem 5** (FPS 2007)**.**
(1) *If $x, y \in J(kE)$ and $x - y \in J^2(kE)$ then $x$ has maximal Jordan type if and only if $y$ does.*
(2) *The points of $J/J^2$ of maximal Jordan type form a dense open subset.*
(3) *This is the same as the Jordan type at a generic point of $\mathbb{A}^r(k)$.*

So we talk of the *generic Jordan type* of $M$.

**Definition 6** (CFP 2008)**.** We say that a $kE$-module $M$ has *constant Jordan type* $[p]^{a_p} \dots [1]^{a_1}$ if every element of $J \setminus J^2$ has this as its Jordan canonical form on $M$. The *stable Jordan type* is $[p-1]^{a_{p-1}} \dots [1]^{a_1}$.

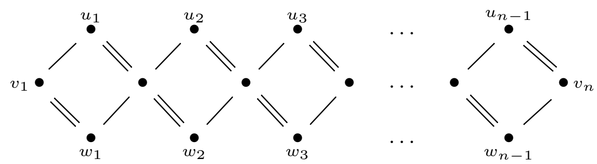
**Example 7.** $E = (\mathbb{Z}/2)^4$, let $M$ be the module

$$aX_1 + bX_2 + cX_3 + dX_4 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 \\ 0 & d & c & b & a & 0 \end{pmatrix}$$




Then $M$ has constant Jordan type $[2]^2\,[1]^2$. The same construction gives constant Jordan type $[2]^2\,[1]^{2n-2}$ for $(\mathbb{Z}/2)^{2n}$.

**Example 8.** Let $E = (\mathbb{Z}/p)^2$, $p \geq 3$, and let $M_n$ $(n \geq 2)$ be the module




Then $M_n$ has constant Jordan type if and only if $n$ is divisible by $p$. The Jordan type is $[3]^{n-2}[2]^2$.

**Question 9.** Suppose that $r \geq 2$. What stable constant Jordan types can occur?

**Lemma 10** (Dade's Lemma, 1978)**.** *If $M$ has constant Jordan type $[p]^n$ then $M$ is a free = projective = injective $kE$-module. In particular $p^{r-1}|n$.*

**Theorem 11** (CFP)**.** *Every summand of a module of constant Jordan type has constant Jordan type.*

**Tensor products and duals:** We make $M \otimes_k N$ into a $kE$-module via $g(m \otimes n) = gm \otimes gn$ We make $M^* = \mathsf{Hom}_k(M, k)$ into a $kE$-module via $g(f)(m) = f(g^{-1}(m))$.

**Theorem 12** (CFP)**.** *If $M$ and $N$ have constant Jordan type then so do $M^*$ and $M \otimes_k N$.*

**Warning 13.** In general $(M \otimes_k N){\downarrow_{X_\alpha}} \not\cong M{\downarrow_{X_\alpha}} \otimes_k N{\downarrow_{X_\alpha}}$.

**Endotrivial modules:** What modules have stable constant Jordan type $[1]$ or $[p-1]$? If $M$ has stable constant Jordan type $[1]$ or $[p-1]$ then $M \otimes_k M^*$ has stable constant Jordan type $[1]$. Then $k \to M \otimes_k M^* \to k$ has non-zero composite since $p \nmid \dim M$. So $M \otimes_k M^* = k \oplus$ a module of constant Jordan type $[p]^n$. So by Dade's lemma this is $= k \oplus$ (free). So $M$ is *endotrivial*. Dade (1978) classified these for $kE$, and $M \cong \Omega^n k$ ($n \in \mathbb{Z}$). These modules do indeed have stable constant Jordan type $[1]$ if $n$ is even, $[p-1]$ if $n$ is odd.

**Single stable Jordan block:** CFP conjectured that there is no module of stable constant Jordan type $[2]$ if $p \geq 5$ and $r \geq 2$. More generally we have the following.

**Theorem 14** (B, MSRI 2008)**.** *If $r \geq 2$ and $2 \leq a \leq p-2$ then there is no module of stable constant Jordan type $[a]$.*

*Proof.* We have $\dim M = np + a$ and so

$$\dim \Lambda^{a+1} M = \frac{(np+a)\dots(np)}{(a+1) \quad \dots \quad 1}.$$

It follows that $\Lambda^{a+1}M$ is free by Dade's lemma, so $p|n$. Similarly we have

$$\dim S^{p-a+1}M = \frac{(np+a)\dots(np+p)}{(p-a+1)\quad\dots 1}$$

Dade's lemma: $S^{p-a+1}M$ is also free so $p|(n+1)$, a contradiction. $\square$

For the last line of this proof, the freeness of $S^{p-a+1}M$ we need the following lemma.

**Lemma 15** (Almkvist & Fossum 1978)**.** *As modules for $k[t]/(t^p)$, $S^i[a]$ is free provided $i < p$, $a + i > p$.*

*Proof.* True if $a = p$. Downward induct on $a$ using

$$0 \to S^i[a] \to S^{i+1}[a] \to S^{i+1}[a-1] \to 0 \qquad \square$$

Here are some conjectures about modules of constant Jordan type.

**Conjecture 16** (Rickard, MSRI 2008)**.** Suppose $r \geq 2$ and $M$ is a $kE$-module of constant Jordan type. If there are no Jordan blocks of length $i$ then the total number of Jordan blocks of length $> i$ is divisible by $p$.

This implies the previous theorem, since there are no Jordan blocks of length $a-1$ or $a+1$. That theorem is also implied by the following conjecture.

**Conjecture 17** (S, in CFP)**.** Let $r \geq 2$. If $2 \leq i \leq p-1$ and $M$ has constant Jordan type with blocks of length $i$, then it also has blocks of length either $i-1$ or $i+1$.

**Conjecture 18** (CFP)**.** Let $r \geq 2$, $p \geq 5$. If there's a module of stable constant Jordan type $[2][1]^j$ then $j \geq r-1$.

**Definition 19** (CF, CFS)**.** A module $M$ has the *constant image property* if for all $0 \neq \alpha \in \mathbb{A}^r(k)$ we have $X_\alpha.M = \mathsf{Rad}(M)$. Equivalently, for all $X \in J(kE) \setminus J^2(kE)$ we have $X.M = \mathsf{Rad}(M)$.

**Lemma 20.** *If $M$ has the constant image property then for all $1 \leq j \leq p$ we have $X_\alpha^j.M = \mathsf{Rad}^j(M)$. In particular, $\mathsf{Rad}^p(M) = 0$.*

**Theorem 21.** *If $M$ has the constant image property then it has constant Jordan type.*

**Definition 22** (CFS)**.** Let $E = \mathbb{Z}/p \times \mathbb{Z}/p$. The *generic kernel* of $M$ is

$$\mathfrak{K}(M) = \bigcap_{\substack{S \subseteq \mathbb{P}^1 \\ \text{cofinite}}} \sum_{\bar{\alpha} \in S} \mathsf{Ker}(X_\alpha, M).$$

**Properties:**

- $\mathfrak{K}(\mathfrak{K}(M)) = \mathfrak{K}(M)$
- $\mathfrak{K}(M)$ has the constant image property, hence constant Jordan type.
- If $N \subseteq M$ has the constant image property then $N \subseteq \mathfrak{K}(M)$.
- $\mathsf{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$ if and only if $X_\alpha$ has maximal rank on $M$.
- So if $M$ has constant Jordan type then for all $\alpha \neq 0$ we have $\mathsf{Ker}(X_\alpha, M) \subseteq \mathfrak{K}(M)$.

**Theorem 23.** *If $M$ has constant Jordan type then $J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ also has constant Jordan type with the same number of Jordan blocks of length one.*

**Theorem 24** (B 2011, special case of Rickard's conjecture)**.** *Let $E = \mathbb{Z}/p \times \mathbb{Z}/p$ and $M$ have constant Jordan type with no Jordan blocks of length $1$. Then the total number of Jordan blocks is divisible by $p$.*

*Idea of Proof.* Show that $\mathfrak{K}(M)/J^2\mathfrak{K}(M)$ is a sum of modules of the form . Then the fact that for all $\alpha \neq 0$ the map $X_\alpha$ from $J^{-1}\mathfrak{K}(M)/\mathfrak{K}(M)$ to $\mathfrak{K}(M)/J(\mathfrak{K}(M))$ is injective is used in order to show that the lengths of the tops of these summands are all divisible by $p$. □

**Corollary 25.** *If $M$ has constant Jordan type with no Jordan blocks of length $p-1$ then the number of Jordan blocks of length $p$ is divisible by $p$.*

*Proof.* Apply the theorem to $\Omega(M)$. □

**Corollary 26.** *If $M$ has constant Jordan type with no Jordan blocks of length $1$ or $p-1$ then the number of Jordan blocks of length between $2$ and $p-2$ is divisible by $p$.*

## 2. Vector Bundles on Projective Space

**Definition 27.** Let $\mathbb{P}^{r-1} = \mathsf{Proj}\, k[Y_1, \dots, Y_r]$ where $Y_1, \dots, Y_r$ have degree one. A *vector bundle* on $\mathbb{P}^{r-1}$ is a locally free sheaf of $\mathcal{O}$-modules, where $\mathcal{O}$ is the structure sheaf on $\mathbb{P}^{r-1}$.

**Theorem 28** (Exercise II.5.9 of Hartshorne)**.** *There is an equivalence of categories between coherent sheaves on $\mathbb{P}^{r-1}$ and finitely generated graded modules over $k[Y_1, \dots, Y_r]$ modulo finite length modules.*

**Twists**: For a graded module $M$ we define $M(j)_i = M_{i+j}$. For sheaves, $\mathcal{O}(1)$ is the *twisting sheaf* generated by global sections $Y_1, \dots, Y_r$. Then the *twists* of a sheaf are given by $\mathcal{F}(j) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(1)^{\otimes j}$.

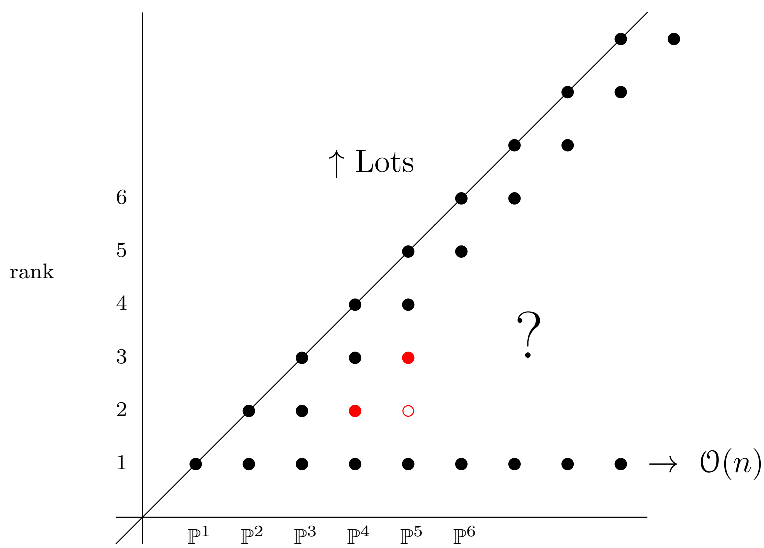
**Remark 29.** The only line bundles on $\mathbb{P}^{r-1}$ are $\mathcal{O}(n)$ for $n \in \mathbb{Z}$.

If $r = 2$ then every vector bundle on $\mathbb{P}^1$ is a sum of line bundles (Grothendieck).

If $r \geq 3$, it is easy to construct indecomposable vector bundles on $\mathbb{P}^{r-1}$ of every rank at least $r-1$. Bundles of rank $r-2$ are slightly more difficult to construct, but examples include the null correlation bundle and instanton bundles ($r$ even), some bundles of Tango (all $r$) and others.

The only known indecomposable vector bundles with rank bigger than 1 and less than $r-2$ are:

- **$\mathbb{P}^4$:** the Horrocks–Mumford bundle $\mathcal{F}_{\mathrm{HM}}$ of rank 2 with 15,000 symmetries,
- **$\mathbb{P}^5$:** Horrocks' Parent bundle of rank 3,
- $\mathbb{P}^5$ in characteristic 2: the Tango bundle of rank 2,
- ... a few more of rank 2 on $\mathbb{P}^4$ and rank 3 on $\mathbb{P}^5$ in char $p$,
- ... and bundles obtained from these in obvious ways.




**Vector Bundles from Modules of constant Jordan type:** Let $\mathbb{P}^{r-1} = \mathsf{Proj}\, k[Y_1, \ldots, Y_r]$ where $Y_i$ are functions on $\mathbb{A}^r$ defined by $Y_i(X_j) = \delta_{ij}$. Given a $kE$-module $M$, set $\widetilde{M} = M \otimes_k \mathcal{O}$, a trivial bundle whose rank is equal to the dimension of $M$.

**Definition 30** (FP, TAMS 2011)**.** We define $\theta \colon \widetilde{M}(j) \to \widetilde{M}(j+1)$ via

$$\theta(m \otimes f) = \sum_i X_i m \otimes Y_i f.$$

**Intuitive idea:** At $\bar{\alpha} = (\lambda_1 : \cdots : \lambda_r) \in \mathbb{P}^{r-1}$ the action of $\theta$ is via

$$m \otimes 1 \mapsto \sum_i X_i m \otimes \lambda_i = \sum_i \lambda_i X_i m \otimes 1 = X_\alpha m \otimes 1.$$

i.e., the action of $\theta$ on the copy of $M$ at $\bar{\alpha} \in \mathbb{P}^{r-1}$ is via $X_\alpha$. Notice that the twist is necessary in order to make this well defined on projective space.

**Definition 31** (BP, MSRI 2008)**.** We define $\mathcal{F}_i(M) = \dfrac{\mathsf{Ker}\theta \cap \mathsf{Im}\theta^{i-1}}{\mathsf{Ker}\theta \cap \mathsf{Im}\theta^{i}}$ as subquotient of $\widetilde{M}$.

Now $\mathsf{Ker}\theta$ picks out the bottoms of all Jordan blocks. $\mathsf{Ker}\theta \cap \mathsf{Im}\theta^i$ picks out the bottoms of the Jordan blocks of length at least $i+1$. So $\mathcal{F}_i$ picks out the bottoms of the Jordan blocks of length $i$. Thus $\mathcal{F}_i(M)$ is a vector bundle iff the number of Jordan blocks of length $i$ is independent of $\bar{\alpha} \in \mathbb{P}^{r-1}$.

**Proposition 32.** *$\mathcal{F}_i(M)$ is a vector bundle for $1 \le i \le p$ if and only if $M$ has contant Jordan type.*

More generally, define

$$\mathcal{F}_{i,j}(M) = \frac{\mathsf{Ker}\theta^{j+1} \cap \mathsf{Im}\theta^{i-j-1}}{(\mathsf{Ker}\theta^{j+1} \cap \mathsf{Im}\theta^{i-j}) + (\mathsf{Ker}\theta^{j} \cap \mathsf{Im}\theta^{i-j-1})}$$

This captures the $(j+1)$st layer from the bottom of the Jordan blocks of length $i$. In particular $\mathcal{F}_{i,0}(M) = \mathcal{F}_i(M)$. The map $\theta$ induces an isomorphism $\mathcal{F}_{i,j}(M) \to \mathcal{F}_{i,j-1}(M)(1)$. Therefore we have

$$\mathcal{F}_{i,j}(M) \cong \mathcal{F}_i(M)(j).$$

**Observation:** $\widetilde{M}$ has a filtration with filtered quotients $\mathcal{F}_{i,j}(M)$ $(0 \le j < i \le p)$:

$$\begin{tikzcd}[column sep=small, row sep=small]
\mathcal{F}_{p,p-1} \arrow[dr, dash] & & & & \\
 & \mathcal{F}_{p-1,p-2} \arrow[dl, dash] \arrow[dr, dash] & & & \\
\mathcal{F}_{p,p-2} \arrow[dr, dash] & & \mathcal{F}_{p-2,p-3} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \mathcal{F}_{p-1,p-3} \arrow[dl, dash] \arrow[dr, dash] & & \cdots \arrow[dl, dash] \arrow[dr, dash] & \\
\cdots \arrow[dr, dash] & \cdots & \cdots \arrow[dl, dash] \arrow[dr, dash] & \cdots & \mathcal{F}_{1,0} \arrow[dl, dash] \\
 & \mathcal{F}_{p-1,1} \arrow[dl, dash] \arrow[dr, dash] & & \cdots \arrow[dl, dash] & \\
\mathcal{F}_{p,1} \arrow[dr, dash] & & \mathcal{F}_{p-2,0} \arrow[dl, dash] & & \\
 & \mathcal{F}_{p-1,0} \arrow[dl, dash] & & & \\
\mathcal{F}_{p,0} & & & &
\end{tikzcd}$$

**Interpretation of $\theta$:** Think of a homomorphism from $\widetilde{M}$ to $\widetilde{M}(1)$ as an $n \times n$ matrix of elements of $\mathsf{Hom}(\mathcal{O}, \mathcal{O}(1))$. Now $\mathsf{Hom}(\mathcal{O}, \mathcal{O}(1))$ is a vector space with basis $Y_1, \dots, Y_r$. So we can think of $\theta$ as being a matrix of linear forms,

$$\sum_i Y_i \phi_M(X_i) \in \mathsf{Mat}_n(k[Y_1, \dots, Y_r])$$

where $\phi_M \colon kE \to \mathsf{Mat}_n(k)$ gives the representation of $E$ on $M$.

**Example 33.** Let $E = (\mathbb{Z}/p)^2 = \langle g_1, g_2 \rangle$, $kE = k[X_1, X_2]/(X_1^p, X_2^p)$, and let $M$ be given by

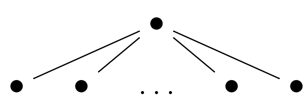
$$g_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g_2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then

$$\theta = Y_1 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + Y_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ Y_1 & 0 & 0 \\ Y_2 & 0 & 0 \end{pmatrix}.$$

The operator $\theta$ has kernel of rank two and image of rank one; $\mathcal{F}_1(M)$ and $\mathcal{F}_2(M)$ are both rank one bundles.

**Example 34.** More generally $M = kE/J^2(kE)$ has constant Jordan type $[2][1]^{r-1}$.



$\theta \colon \widetilde{M} \to \widetilde{M}(1)$ has $\widetilde{\mathsf{Soc}(M)}$ in its kernel, and its image is $\mathcal{F}_2(M)(1) \subseteq \widetilde{M}(1)$. So $\mathcal{O} \cong \widetilde{M/\mathsf{Soc}(M)} \xrightarrow{\theta} \mathcal{F}_2(M)(1)$ gives $\mathcal{F}_2(M) \cong \mathcal{O}(-1)$. On the other hand, $\mathcal{F}_1(M)$ is $\widetilde{\mathsf{Soc}(M)}/\mathcal{F}_2(M)$ so

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus r} \to \mathcal{F}_1(M) \to 0.$$

The left hand map has coordinates $Y_1, \dots, Y_r$ so this is a twisted version of the *Euler sequence* defining the *tangent bundle*

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus r} \to \mathcal{T} \to 0.$$

Thus we have

$$\mathcal{F}_1(M) \cong \mathcal{T}(-1), \qquad \mathcal{F}_2(M) \cong \mathcal{O}(-1).$$

**Example 35.** The module $M = \mathsf{Soc}^2(kE)$ also has constant Jordan type $[2][1]^{r-1}$. But this time $\mathcal{F}_1(M) \cong \Omega(1)$, $\mathcal{F}_2(M) \cong \mathcal{O}$ where $\Omega$ is the cotangent bundle.

**Example 36** (B, MSRI 2008)**.** If $p \geq 7$, $r = 5$, there exists a $kE$-module $M$ such that $\mathcal{F}_2(M) \cong \mathcal{F}_{\mathrm{HM}}(-2)$. We have $\dim M = 30p^5$, and $M$ has stable constant Jordan type $[p-1]^{30}[2]^2[1]^{26}$.

**Example 37.** The following is an example with $E = (\mathbb{Z}/2)^6$, where $\mathcal{F}_1(M)$ is the rank two Tango bundle on $\mathbb{P}^5$.

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 & & & & & & & & & & & f & & d & b & & & a & & c & & & & & & & & & & \\
 & & & & & & & & & & & e & d & & c & & & & c & & a & b & & & & & & & & \\
 & & & & & & & & & c & b & & & & & & & e & d & & f & & & & & & & & & \\
 & & & & & & & & c & & a & & & & e & d & & & & & & f & & & & & & & & \\
 & & & & & & & & b & a & & & & & f & & d & & f & e & & & & & & & & & & \\
\hline
 & & & & & & & & d & e & f & a & b & c & & & & & & & & & & & & & & & & \\
\hline\hline
 & & & & & & & & & & & & & & & & & & & & & & d & e & f & a & b & c & & 
\end{array}\right)
$$

**Properties of $\mathcal{F}_i$:**

$$
\begin{aligned}
&\mathcal{F}_{p-i}(\Omega M) \cong \mathcal{F}_i(M)(-p+i) \qquad (1 \leq i \leq p-1)\\
&\mathcal{F}_i(M^*) \cong \mathcal{F}_i(M)^\vee(-i+1) \qquad (1 \leq i \leq p)\\
&\mathcal{F}_1(M \otimes_k N) \cong \bigoplus_{i=1}^{p-1} \mathcal{F}_i(M) \otimes_{\mathcal{O}} \mathcal{F}_i(N)(i-1)
\end{aligned}
$$

The sequence $0 \to \Omega M \to P_M \to M \to 0$ induces

$$0 \to \mathcal{F}_p(\Omega M) \to \mathcal{F}_p(P_M) \to \mathcal{F}_p(M) \to 0.$$

This is not exact, but has homology only in the middle, where it is

$$\bigoplus_{i=1}^{p-1} \mathcal{F}_i(M)(-p+i).$$

**Theorem 38** (Realisation Theorem (BP, MSRI 2008))**.** *Given a vector bundle $\mathcal{F}$ of rank $s$ on $\mathbb{P}^{r-1}$, there exists a $kE$-module $M$ of stable constant Jordan type $[1]^s$ such that*

- *if $p = 2$ then $\mathcal{F}_1(M) \cong \mathcal{F}$*
- *if $p$ is odd then $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$*

*where $F\colon \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the Frobenius map.*

Let us outline the proof of the realisation theorem. We begin with $p = 2$. Given $\mathcal{F}$, Hilbert's syzygy theorem gives a resolution

$$0 \to \sum_{j=1}^{m_r} \mathcal{O}(a_{r,j}) \to \cdots \to \sum_{j=1}^{m_1} \mathcal{O}(a_{1,j}) \to \sum_{j=1}^{m_0} \mathcal{O}(a_{0,j}) \to \mathcal{F} \to 0.$$

If $a > b$ then $\mathsf{Hom}(\mathcal{O}(a), \mathcal{O}(b)) = 0$, while if $a \leq b$ it is the space of degree $b - a$ polynomials in $Y_1, \ldots, Y_r$. Now mimic this with representations of $kE$. We have

$$H^*(E, k) = k[y_1, \ldots, y_r].$$

We have $\mathcal{F}_1(\Omega^{-a}(k)) \cong \mathcal{O}(a)$ and provided $a \leq b$

$$\underline{\mathsf{Hom}}_{kE}(\Omega^{-a}(k), \Omega^{-b}(k)) \cong H^{b-a}(E, k)$$

is the space of degree $b - a$ polynomials in $y_1, \ldots, y_r$.

**Lemma 39.** *Representatives $\hat{y}_i\colon \Omega^{n+1}(k) \to \Omega^n(k)$ of $y_i \in H^1(E, k)$ can be found so that $\hat{y}_i\hat{y}_j = \hat{y}_j\hat{y}_i\colon \Omega^{n+2}(k) \to \Omega^n(k)$.*

Define a $k$-algebra homomorphism

$$\rho\colon H^*(E, k) = k[y_1, \ldots, y_r] \to k[Y_1, \ldots, Y_r]$$

by $\rho(y_i) = Y_i$. Then representing an element $\zeta \in H^*(E, k)$ by a cocycle $\hat{\zeta}\colon \Omega^{n+j}(k) \to \Omega^j(k)$ the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(-n-j) \arrow[r, "\rho(\zeta)"] \arrow[d, "\cong"] & \mathcal{O}(-j) \arrow[d, "\cong"] \\
\mathcal{F}_1(\Omega^{n+j}(k)) \arrow[r, "\mathcal{F}_1(\hat{\zeta})"] & \mathcal{F}_1(\Omega^j(k))
\end{tikzcd}$$

Now take a resolution of $\mathcal{F}$

$$0 \to \sum_{j=1}^{m_r} \mathcal{O}(a_{r,j}) \to \cdots \to \sum_{j=1}^{m_1} \mathcal{O}(a_{1,j}) \to \sum_{j=1}^{m_0} \mathcal{O}(a_{0,j}) \to \mathcal{F} \to 0.$$

Apply $\rho^{-1}$ to the entries in the maps in this complex to get

$$0 \to \sum_{j=1}^{m_r} \Omega^{-a_{r,j}}(k) \to \cdots \to \sum_{j=1}^{m_1} \Omega^{-a_{1,j}}(k) \to \sum_{j=1}^{m_0} \Omega^{-a_{0,j}}(k) \to 0.$$

Rickard has a "totalisation" functor $\mathsf{D}^b(kE) \to \mathsf{stmod}(kE)$. Applying this to the above complex gives a module $M$ with $\mathcal{F}_1(M) \cong \mathcal{F}$.

When $p$ is odd, the best we can do is to get a module of type $[p]^a[1]^b$ with $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$. We'll see using Chern classes why this is best possible. We have

$$H^*(E,k) \cong \Lambda(y_1,\dots,y_r) \otimes k[x_1,\dots,x_r]$$

with $\deg(y_i) = 1$, $\deg(x_i) = 2$.

**Lemma 40.** *Representatives $\hat{x}_i \colon \Omega^{n+2}(k) \to \Omega^n(k)$ of $x_i \in H^2(E,k)$ can be found so that $\hat{x}_i\hat{x}_j = \hat{x}_j\hat{x}_i \colon \Omega^{n+4}(k) \to \Omega^n(k)$.*

We have $\mathcal{F}_1(\Omega^{-2a}(k)) \cong \mathcal{O}(pa)$. Define a $k$-algebra homomorphism

$$\rho\colon k[x_1,\dots,x_r] \to k[Y_1,\dots,Y_r]$$

by $\rho(x_i) = Y_i^p$. Then representing an element $\zeta \in k[x_1,\dots,x_r]$ by $\hat{\zeta}\colon \Omega^{n+j}(k) \to \Omega^j(k)$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{O}(-p(n+j)) \arrow[r, "\rho(\zeta)"] \arrow[d, "\cong"] & \mathcal{O}(-pj) \arrow[d, "\cong"] \\ \mathcal{F}_1(\Omega^{2(n+j)}(k)) \arrow[r, "\mathcal{F}_1(\hat{\zeta})"] & \mathcal{F}_1(\Omega^{2j}(k)) \end{tikzcd}$$

Now take a resolution of $\mathcal{F}$

$$0 \to \sum_{j=1}^{m_r} \mathcal{O}(a_{r,j}) \to \cdots \to \sum_{j=1}^{m_1} \mathcal{O}(a_{1,j}) \to \sum_{j=1}^{m_0} \mathcal{O}(a_{0,j}) \to \mathcal{F} \to 0.$$

Each map is a matrix of polynomials in $Y_1,\dots,Y_r$. Replace each $Y_i$ by $Y_i^p$ to get a complex

$$0 \to \sum_{j=1}^{m_r} \mathcal{O}(pa_{r,j}) \to \cdots \to \sum_{j=1}^{m_1} \mathcal{O}(pa_{1,j}) \to \sum_{j=1}^{m_0} \mathcal{O}(pa_{0,j}) \to F^*(\mathcal{F}) \to 0.$$

This is a resolution of $F^*(\mathcal{F})$, where $F\colon \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the Frobenius map induced by

$$\begin{aligned} k[Y_1,\dots,Y_r] &\to k[Y_1,\dots,Y_r] \\ Y_i &\mapsto Y_i^p \end{aligned}$$

The entries in the maps are now in the image of $\rho$. Apply $\rho^{-1}$ to the entries to get a complex

$$0 \to \sum_{j=1}^{m_r} \Omega^{-2a_{r,j}}(k) \to \cdots \to \sum_{j=1}^{m_1} \Omega^{-2a_{1,j}}(k) \to \sum_{j=1}^{m_0} \Omega^{-2a_{0,j}}(k) \to 0.$$

Again apply Rickard's totalisation functor $\mathsf{D}^b(kE) \to \mathsf{stmod}(kE)$ to the above complex to get a module $M$ with $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$.

## 3. Chern Classes

The *Chow group* $A^*(\mathbb{P}^{r-1})$ is isomorphic to $\mathbb{Z}[h]/(h^r)$. Given a vector bundle $\mathcal{F}$ on $\mathbb{P}^{r-1}$, there is a *Chern polynomial*

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F})h + \cdots + c_{r-1}(\mathcal{F})h^{r-1} \in A^*(\mathbb{P}^{r-1})$$

whose coefficients $c_i(\mathcal{F})$ are the *Chern numbers* of $\mathcal{F}$. We'll construct the Chern polynomial in this lecture without reference to the general definition of Chow group. Recall

**Theorem 41** (Exercise II.5.9 of Hartshorne)**.** *There is an equivalence of categories between coherent sheaves on $\mathbb{P}^{r-1} = \mathsf{Proj}\, k[Y_1, \ldots, Y_r]$ and finitely generated graded modules over $k[Y_1, \ldots, Y_r]$ modulo finite length modules.*

**Definition 42.** If $M = \bigoplus_{j\in\mathbb{Z}} M_j$ is a finitely generated graded module over $R = k[Y_1, \ldots, Y_r]$ then the *Poincaré series* (or *Hilbert series*) of $M$ is

$$p_M(t) = \sum_{j\in\mathbb{Z}} t^j \dim_k M_j.$$

**Lemma 43** (Hilbert, Serre)**.** *The Poincaré series of a finitely generated $R$-module takes the form $p_M(t) = \dfrac{f(t)}{(1-t)^r}$ where $f(t)$ is a Laurent polynomial.*

*If $M$ is a finite length module then $p_M(t)$ is a Laurent polynomial. i.e., $f(t)$ is divisible by $(1-t)^r$.*

**Definition 44.** We define the *rank* of $M$ to be the positive integer $f(1)$. This is equal to the dimension of the ungraded vector space

$$k(Y_1, \ldots, Y_r) \otimes_{k[Y_1,\ldots,Y_r]} M$$

over the field $k(Y_1, \ldots, Y_r)$.

**Lemma 45.** *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of graded $R$-modules then $p_{M_2}(t) = p_{M_1}(t) + p_{M_3}(t)$.*

If $M$ corresponds to a vector bundle $\mathcal{F}$ on $\mathbb{P}^{r-1}$ then the rank is the dimension of the vector space at each point.

**Definition 46.** We define the Chow ring of $R$ to be the truncated polynomial ring $A^*(R) = \mathbb{Z}[h]/(h^r)$.

**Definition 47.** If $p_M(t) = \dfrac{\sum_j a_j t^j}{(1-t)^r}$ then we define the *Chern polynomial* of $M$ to be

$$c(M) = \prod_j (1+jh)^{a_j} \in A^*(R) = \mathbb{Z}[h]/(h^r).$$

Some $a_j$ may be negative, but $(1+jh)$ is invertible in $A^*(R)$.

The *Chern numbers* of $M$ are the coefficients

$$c(M) = 1 + c_1(M)h + \cdots + c_{r-1}(M)h^{r-1}$$

and by convention $c_0(M) = 1$.

The *Chern character* of $M$ is defined to be

$$\mathsf{Ch}(M) = \sum_j a_j e^{jh} \in A^*_{\mathbb{Q}}(R) = \mathbb{Q} \otimes_{\mathbb{Z}} A^*(R) = \mathbb{Q}[h]/(h^r).$$

$$\mathsf{Ch}(M) = \mathrm{rank}\,(M) + c_1 h + \tfrac{1}{2}(c_1^2 - 2c_2)h^2 + \cdots$$

**Lemma 48.** *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence then*

(i) $c(M_2) = c(M_1)c(M_3)$ — *i.e.,* $c_j(M_2) = \sum_{i=0}^{j} c_i(M_1)c_{j-i}(M_3)$
(ii) $\mathsf{Ch}(M_2) = \mathsf{Ch}(M_1) + \mathsf{Ch}(M_3)$.

**Lemma 49.** *If $M$ and $M'$ are equivalent modulo finite length modules then $c(M) = c(M')$ and $\mathsf{Ch}(M) = \mathsf{Ch}(M')$.*

*Proof.* For $\mathsf{Ch}(M)$, easy: $\mathsf{Ch}(k[n]) = \sum_j (-1)^j \binom{r}{j} e^{(j+n)h} = e^{nh}(1-e^h)^r$ and $1 - e^h$ is divisible by $h$.
For $c(M)$, need $c(k[n]) = 1$. This follows from:

$$c(k[n]) = \prod_{j=0}^{r} (1+(j+n)h)^{(-1)^j\binom{r}{j}} \equiv 1 \pmod{h^r}. \qquad \square$$

**Definition 50.** If a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{r-1}$ corresponds to a finitely generated graded $k[Y_1,\dots,Y_r]$-module $M$ then we define $c(\mathcal{F}) = c(M)$ and $\mathsf{Ch}(\mathcal{F}) = \mathsf{Ch}(M)$.

**Exercise:** Show that $c(\mathcal{F}(1), h) = (1+h)^{\mathrm{rank}\,\mathcal{F}} c\left(\mathcal{F}, \frac{h}{1+h}\right)$.

**Fact:** For a vector bundle $c_i(\mathcal{F}) = 0$ for $i > \mathrm{rank}\,\mathcal{F}$.

Next we discuss congruences on Chern classes.

**Lemma 51.** *For a vector bundle $\mathcal{F}$ of rank $s$ in $\mathbb{Z}[h]/(h^r)$ we have*

$$c(\mathcal{F})c(\mathcal{F}(1))\dots c(\mathcal{F}(p-1)) \equiv 1 - sh^{p-1} \pmod{p, h^{2p-2}}.$$

*Proof.* Recall that if $p_M(t) = \sum_j a_j t^j/(1-t)^r$ then $c(\mathcal{F}) = \prod_j (1+jh)^{a_j}$.

$$c(\mathcal{F})\dots c(\mathcal{F}(p-1)) = \prod_j ((1+jh)\dots(1+(j+p-1)h))^{a_j}.$$

Since $x(x+y)(x+2y)\dots(x+(p-1)y) \equiv x^p - xy^{p-1} \pmod{p}$, this

$$\begin{aligned}
&\equiv \prod_j ((1+jh)^p - (1+jh)h^{p-1})^{a_j} \pmod{p}\\
&\equiv \prod_j (1-h^{p-1}+(j^p-j)h^p)^{a_j} \pmod{p}\\
&\equiv \prod_j (1-h^{p-1})^{a_j} \pmod{p}\\
&\equiv 1-\sum_j a_j h^{p-1} \pmod{p, h^{2p-2}}.
\end{aligned}$$
□

**Theorem 52** (BP)**.** *Suppose $r \geq 2$, and let $M$ be a $kE$-module of stable constant Jordan type $[1]^s$. Then $p|c_i(\mathcal{F}_1(M))$ for $1 \leq i \leq p-2$.*

*Proof.* $\widetilde{M}$ has a filtration with filtered quotients

$$\mathcal{F}_1(M),\ \mathcal{F}_p(M),\ \mathcal{F}_p(M)(1), \dots, \mathcal{F}_p(M)(p-1).$$

Therefore $c(\mathcal{F}_1(M))c(\mathcal{F}_p(M))c(\mathcal{F}_p(M)(1))\dots c(\mathcal{F}_p(M)(p-1)) = 1$. By the lemma, it follows that $c(\mathcal{F}_1(M)) \equiv 1 + sh^{p-1} \pmod{p, h^{2p-2}}$. □

If $p = 2$ this gives no information, but for $p$ odd it gives a genuine restriction on the vector bundles that can occur this way. In particular, it throws light on the realisation theorem.

**Remark 53.** If $F$ is the Frobenius map then $c(F^*(\mathcal{F}), h) = c(\mathcal{F}, ph)$. So the condition is satisfied by $F^*(\mathcal{F})$.

**Example 54.** The rank two Horrocks–Mumford bundle $\mathcal{F}_{\mathrm{HM}}$ on $\mathbb{P}^4$ has $c_1(\mathcal{F}_{\mathrm{HM}}(i)) = 2i+5$ and $c_2(\mathcal{F}_{\mathrm{HM}}(i)) = i^2+5i+10$. So no twist of $\mathcal{F}_{\mathrm{HM}}$ can occur as $\mathcal{F}_1(M)$ for a module of stable constant Jordan type $[1]^2$. But by the realisation theorem there is a module $M$ of stable constant Jordan type $[1]^2$ with $\mathcal{F}_1(M) \cong F^*(\mathcal{F}_{\mathrm{HM}})$.

A similar analysis of Chern classes shows that for large rank and large primes, the only small stable constant Jordan type is $[1]^s$:

**Theorem 55** (B, 2010)**.** *If a module has stable constant Jordan type $[a_1][a_2]\dots[a_t]$ with $a = \sum a_i \leq \min(r-1, p-2)$ then $a_1 = \dots = a_t = 1$.*

*Proof.* We have

$$\prod_{j=1}^{p-2} c(\mathcal{F}_j(M))c(\mathcal{F}_j(M)(1))\dots c(\mathcal{F}_j(M)(j-1)) \equiv 1 \pmod{(p, h^{p-1})}.$$

This is a polynomial of degree $a \le p-2$. Also $a \le r-1$ so this can be read as an equality in $\mathcal{F}_p[h]$. The only units in this ring are the constants. So for $j \ge 2$ both $c(\mathcal{F}_j(M))$ and $c(\mathcal{F}_j(M)(1))$ have to be 1 mod $p$. But $c_1(\mathcal{F}_j(M)(1)) = c_1(\mathcal{F}_j(M)) + \operatorname{rank}\mathcal{F}_j(M)$ so $p|\operatorname{rank}\mathcal{F}_j(M)$, a contradiction. □

This proves a weak form of the conjecture of CFP. Recall:

**Conjecture 56.** Let $r \ge 2$, $p \ge 5$. If $M$ has stable constant Jordan type $[2][1]^j$ then $j \ge r-1$.

**Corollary 57.** *If $M$ has stable constant Jordan type $[2][1]^j$ and $p \ge j+4$ then $j \ge r-2$.*

The smallest case where there's a discrepancy between the conjecture and the corollary is type $[2][1]$ for $r = 3$, $p \ge 5$. In this case it can be proved that $p \equiv 1 \pmod 3$.

**Chern roots:** The Chern polynomial $c(\mathcal{F}) \in \mathbb{Z}[h]/(h^r)$ has a unique lift to $\mathbb{Z}[h]$ of degree $\le r-1$, also denoted $c(\mathcal{F})$. Factorise it in $\mathbb{C}[h]$: $c(\mathcal{F}) = \prod_j (1 + \alpha_j h)$. The algebraic integers $\alpha_j$ are the *Chern roots* of $\mathcal{F}$.

$$c_1 = \sum_i \alpha_i \qquad c_2 = \sum_{i<j} \alpha_i\alpha_j \qquad \dots$$

The *number* of Chern roots for a coherent sheaf is not well defined, but for a vector bundle it can be taken as the rank.

**Definition 58** (Power sums)**.** $s(\mathcal{F}, h) \in \mathbb{Z}[h]/(h^r)$ is defined by

$$-s(\mathcal{F}, -h) = \frac{hc'(\mathcal{F}, h)}{c(\mathcal{F}, h)} \qquad s(\mathcal{F}, h) = s_1(\mathcal{F})h + s_2(\mathcal{F})h^2 + \dots$$

We have $s_1 = c_1$, $s_2 = c_1^2 - 2c_2$, ...

**Calculation:** $s_n(\mathcal{F}) = \sum_j a_j j^n = \sum_j \alpha_j^n$.

**Theorem 59.** *If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ then $s_n(\mathcal{F}_2) = s_n(\mathcal{F}_1) + s_n(\mathcal{F}_3)$.*

*Proof.* This follows from $s_n(\mathcal{F}) = \sum_j a_j j^n$. □

**Theorem 60.** *If $f(x)$ is any polynomial of degree at most $r-1$ with $f(0) = 0$ then*

$$\sum_j f(\alpha_j) = \sum_j a_j f(j).$$

*Proof.* True for $f(x) = x^n$ for $1 \le n \le r-1$ by previous frame. $\square$

**Consequence (Schwarzenberger's conditions):** If $n \in \mathbb{Z} \Rightarrow f(n) \in \mathbb{Z}$ then $\sum_j f(\alpha_j) \in \mathbb{Z}$.

Examples of such polynomials are binomials $f(n) = \binom{n}{i}$.

For example, on $\mathbb{P}^3$ we have $c_1c_2 + c_3 \equiv 0 \pmod 2$.

For a rank two bundle on $\mathbb{P}^4$ we have $c_2(c_2+1-3c_1-2c_1^2) \equiv 0 \pmod{12}$.

**Theorem 61.** $\mathsf{Ch}(\mathcal{F}) = \operatorname{rank}\mathcal{F} + \sum_j(e^{\alpha_j h} - 1)$.

**Remark 62.** If we assume that the number of Chern roots is the rank of $\mathcal{F}$ this reads as $\mathsf{Ch}(\mathcal{F}) = \sum_j e^{\alpha_j h}$.

*Proof.* We have

$$\begin{aligned}\mathsf{Ch}(\mathcal{F}) = \sum_j a_j e^{jh} &= \operatorname{rank}\mathcal{F} + \sum_j a_j(e^{jh} - 1)\\ &= \operatorname{rank}\mathcal{F} + \sum_j \sum_{n=1}^{r-1} \frac{a_j j^n h^n}{n!}.\end{aligned}$$

Apply Theorem 60:

$$\mathsf{Ch}(\mathcal{F}) = \operatorname{rank}\mathcal{F} + \sum_j \sum_{n=1}^{r-1} \frac{\alpha_j^n h^n}{n!} = \operatorname{rank}\mathcal{F} + \sum_j (e^{\alpha_j h} - 1). \qquad \square$$

**Cohomology of sheaves:** The global section functor $\mathcal{F} \mapsto \Gamma(\mathcal{F}) = \Gamma(\mathbb{P}^{r-1}, \mathcal{F})$ is left exact but not right exact. So it has right derived functors $H^i(\mathcal{F}) = H^i(\mathbb{P}^{r-1}, \mathcal{F})$. e.g. $H^0(\mathcal{F}) = \Gamma(\mathcal{F})$. These vanish for $i \ge r$. A short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ gives

$$0 \to H^0(\mathcal{F}_1) \to H^0(\mathcal{F}_2) \to H^0(\mathcal{F}_3) \to H^1(\mathcal{F}_0) \to \cdots \to H^{r-1}(\mathcal{F}_3) \to 0.$$

**Definition 63.** The *Euler characteristic* of $\mathcal{F}$ is

$$\chi(\mathcal{F}) = \sum_{i=0}^{r-1} (-1)^i \dim H^i(\mathcal{F}).$$

**Lemma 64.** *If* $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ *then* $\chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3)$.

**Theorem 65** (Schwarzenberger)**.** *If* $\mathcal{F}$ *is a coherent sheaf on* $\mathbb{P}^{r-1}$ *then*

$$\chi(\mathcal{F}) = \operatorname{rank}\mathcal{F} + \sum_j \left( \binom{\alpha_j + r - 1}{r-1} - 1 \right).$$

*Proof.* Both sides are additive over short exact sequences. Every coherent sheaf has a finite resolution by sums of line bundles. So it suffices to prove the theorem for $\mathcal{F} = \mathcal{O}(j)$. Serre calculated $H^i(\mathcal{O}(j))$: its dimension is $\binom{r+j-1}{r-1}$ if $i = 0$ $j \geq 0$, $\binom{-j-1}{-r-j}$ if $i = r-1$, $j \leq -r$, zero otherwise. So it is true by direct calculation for $\mathcal{O}(j)$. □

**Theorem 66** (Hirzebruch-Riemann-Roch)**.** *The Euler characteristic* $\chi(\mathcal{F})$ *is the coefficient of* $h^{r-1}$ *in* $\left(\dfrac{h}{1-e^{-h}}\right)^r \mathsf{Ch}(\mathcal{F})$.

**Remark 67.** The expression $\left(\dfrac{h}{1-e^{-h}}\right)^r$ is the *Todd class* of the tangent bundle of $\mathbb{P}^{r-1}$.

**Remark 68.** For simplicity let's assume that the number of Chern roots is rank $\mathcal{F}$ so that $\mathsf{Ch}(\mathcal{F}) = \sum_j e^{\alpha_j h}$.

*Proof.* Cauchy's integral formula: coefficient of $h^{r-1}$ is

$$\frac{1}{2\pi i}\oint \left(\frac{h}{1-e^{-h}}\right)^r \mathsf{Ch}(\mathcal{F})\frac{dh}{h^r} = \sum_j \frac{1}{2\pi i}\oint \frac{e^{\alpha_j h}}{(1-e^{-h})^r}\,dh$$

Substitute $z = 1 - e^{-h}$, $dh = dz/(1-z)$, $e^{\alpha_j h} = 1/(1-z)^{\alpha_j}$

$$\begin{aligned} &= \sum_j \frac{1}{2\pi i}\oint \frac{dz}{z^r(1-z)^{\alpha_j+1}} \\ &= \sum_j \frac{1}{2\pi i}\oint z^{-r}\left(1 + (\alpha_j+1)z + \binom{\alpha_j+2}{2}z^2 + \dots\right)dz \\ &= \sum_j \binom{\alpha_j+r-1}{r-1} = \chi(\mathcal{F}). \end{aligned}$$

□

The following theorem is a typical example of an application of the Hirzebruch–Riemann–Roch theorem to modules of constant Jordan type for $p = 2$.

**Theorem 69** (B, 2010)**.** *Let* $k$ *have char* 2*. If* $M$ *has constant Jordan type* $[2]^n[1]^m$ *with* $m \leq r-3$ *then one of the following occurs:*

(i) $n$ *is congruent to one of* $0, -1, \dots, -m$ *modulo* $2^{r-1}$*, or*
(ii) $r \leq 6$*, or*
(iii) *there is a new vector bundle of low rank on projective space of dimension at least six.*

Let's see why this is. Suppose $M$ has constant Jordan type $[2]^n[1]^m$. The sheaf $\widetilde{M}$ has a filtration with quotients $\mathcal{F}_2(M)$, $\mathcal{F}_2(M)(1)$ and

$\mathcal{F}_1(M)$. So if $\mathcal{F}_1(M)$ is a sum of line bundles $\mathcal{O}(a_1), \ldots, \mathcal{O}(a_m)$ then

$$(1+e^h)\mathsf{Ch}(\mathcal{F}_2(M)) + e^{a_1 h} + \cdots + e^{a_m h} = \mathsf{Ch}(\widetilde{M}) = 2n + m.$$

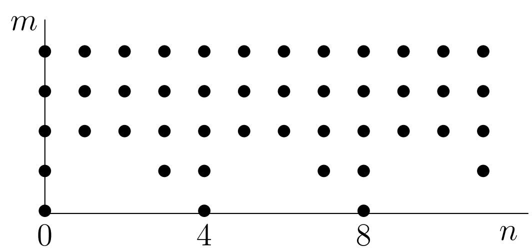
So by the Hirzebruch–Riemann–Roch theorem $\chi(\mathcal{F}_2(M))$ is the coefficient of $h^{r-1}$ in

$$\left(\frac{h}{1-e^{-h}}\right)^r \left(\frac{2n+m-e^{a_1 h} - \cdots - e^{a_m h}}{1+e^h}\right).$$

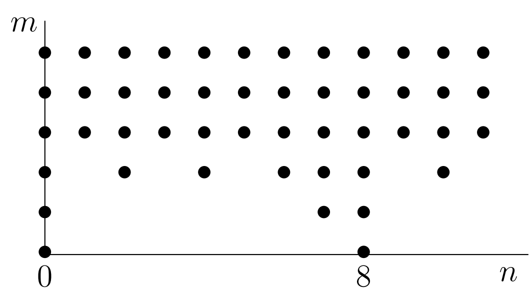
**Calculation:** This differs by an integer from $\dfrac{2n+m-\sum_{i=1}^m (-1)^{a_i}}{2^r}$. So $n$ is congruent mod $2^{r-1}$ to minus the number of odd $a_i$.

**Example 70.** The Tango example of type $[2]^{14}[1]^2$ and rank 6 shows why we need (ii).

**Possible Jordan types for $(\mathbb{Z}/2)^3$: $[2]^n[1]^m$**




**Possible Jordan types for $(\mathbb{Z}/2)^4$: $[2]^n[1]^m$**




**Nilvarieties of constant Jordan type**

**Definition 71.** A *nilvariety* of constant Jordan type $\mathbf{t}$ and rank $r$ consists of nilpotent matrices $A_1, \ldots, A_r$ such that for all $0 \neq \alpha = (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r(k)$ the Jordan canonical form of $\lambda_1 A_1 + \cdots + \lambda_r A_r$ is $\mathbf{t}$.

**Example 72.** The matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

span a nilvariety, but do not commute. For all values of $Y_1$ and $Y_2$, not both zero, the matrix $Y_1A_1 + Y_2A_2$ has a single Jordan block with eigenvalue zero.

**Theorem 73.** *Let $p \geq 3$ and let $M$ be a rank $r$ nilvariety of constant Jordan type $[p]^n$. Then*

$$p^{\left\lfloor \frac{r-1}{p-1} \right\rfloor} | n$$

*where $\left\lfloor \frac{r-1}{p-1} \right\rfloor$ denotes the largest integer less than or equal to $\frac{r-1}{p-1}$.*

*Proof.* Hirzebruch–Riemann–Roch. □

This is sharp for $p = 3$ in characteristic 3, because of tensor products of copies of the previous example.

What happens for $p \geq 5$?

### Nilvarieties with a single Jordan block

**Theorem 74** (Causa, Re and Teodorescu)**.** *Let $M$ be a nilvariety of rank $r$ and constant Jordan type $[m]$. Then $r \leq 2$, and if $r = 2$ then $m$ is odd.*

*Proof.* Suppose that $M$ is a nilvariety of constant Jordan type $[m]$. Then $\mathcal{F}_m(M)$ is a line bundle, so we have $\mathcal{F}_m(M) \cong \mathcal{O}(a)$ for some integer $a$. The bundle $\widetilde{M} \cong \mathcal{O}^{\oplus m}$ has a filtration with filtered quotients $\mathcal{O}(a), \mathcal{O}(a+1), \dots, \mathcal{O}(a+m-1)$ and so

$$0 = c_1(\widetilde{M}) = a + (a+1) + \cdots + (a+m-1) = ma + m(m-1)/2.$$

Thus $a = -(m-1)/2$ and so $m$ is odd. If $r \geq 3$ then for all $i$ and $j$ we have $\mathrm{Ext}^1(\mathcal{O}(i), \mathcal{O}(j)) \cong H^1(\mathbb{P}^{r-1}, \mathcal{O}(j-i)) = 0$ so the filtration splits, giving $\mathcal{O}^{\oplus m} \cong \mathcal{O}(a) \oplus \mathcal{O}(a+1) \oplus \cdots \oplus \mathcal{O}(a+m-1)$, which contradicts the Krull–Schmidt Theorem for vector bundles. □

## Summary Sheet

- $k = \bar{k}$ is a field of characteristic $p$.
- $E = \langle g_1, \ldots, g_r \rangle \cong (\mathbb{Z}/p)^r$
- $X_i = g_i - 1 \in kE$, $X_i^p = 0$.
- $kE = k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)$
- If $\alpha = (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r(k)$ then $X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r \in kE$.
- Generic kernel: $\displaystyle \mathfrak{K}(M) = \bigcap_{\substack{S \subseteq \mathbb{P}^1 \\ cofinite}} \sum_{\bar{\alpha} \in S} \mathsf{Ker}(X_\alpha, M)$
- $\mathbb{P}^{r-1} = \mathsf{Proj}\, k[Y_1, \ldots, Y_r]$, $Y_i(X_j) = \delta_{ij}$
- $\widetilde{M} = M \otimes_k \mathcal{O}$
- $\theta \colon \widetilde{M}(j) \to \widetilde{M}(j+1)$
- $\theta(m \otimes f) = \sum_i X_i m \otimes Y_i f$
- $\displaystyle \mathcal{F}_i(M) = \frac{\mathsf{Ker}\theta \cap \mathsf{Im}\theta^{i-1}}{\mathsf{Ker}\theta \cap \mathsf{Im}\theta^{i}}$ as subquotient of $\widetilde{M}$.
- $\displaystyle \mathcal{F}_{i,j}(M) = \frac{\mathsf{Ker}\theta^{j+1} \cap \mathsf{Im}\theta^{i-j-1}}{(\mathsf{Ker}\theta^{j+1} \cap \mathsf{Im}\theta^{i-j}) + (\mathsf{Ker}\theta^{j} \cap \mathsf{Im}\theta^{i-j-1})}$.
- $\mathcal{F}_{i,j}(M) \cong \mathcal{F}_i(M)(j)$.
- Euler sequence: $0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus r} \to \mathcal{T} \to 0$.
- Chow group $A^*(\mathbb{P}^{r-1}) \cong \mathbb{Z}[h]/(h^r)$.
- Chern polynomial: if $\displaystyle p_M(t) = \frac{\sum_j a_j t^j}{(1-t)^r}$ then

$$c(M) = 1 + c_1(M)h + \cdots + c_{r-1}(M)h^{r-1} = \prod_j (1+jh)^{a_j} \in \mathbb{Z}[h]/(h^r).$$

- Chern character: $\mathsf{Ch}(M) = \sum_j a_j e^{jh} \in \mathbb{Q}[h]/(h^r)$.
- Twists: $c(\mathcal{F}(1), h) = (1+h)^{\operatorname{rank}\mathcal{F}} c(\mathcal{F}, \frac{h}{1+h})$.
- $c(\mathcal{F})c(\mathcal{F}(1)) \ldots c(\mathcal{F}(p-1)) \equiv 1 - (\operatorname{rank}\mathcal{F})h^{p-1} \pmod{p, h^{2p-2}}$.
- Power sums: $-s(\mathcal{F}, -h) = \frac{h}{c'(\mathcal{F},h)} c(\mathcal{F}, h)$
- $s_n(\mathcal{F}) = \sum_j a_j j^n = \sum_j \alpha_j^n$ for $1 \le n < r$.
- Schwarzenberger: $\sum_j \binom{\alpha_j + s}{m} \in \mathbb{Z}$.
- Euler characteristic: $\chi(\mathcal{F}) = \sum_i (-1)^i \dim H^i(\mathbb{P}^{r-1}, \mathcal{F})$.
- Hirzebruch–Riemann–Roch: $\displaystyle \chi(\mathcal{F}) = \sum_j \binom{\alpha_j + r - 1}{r-1}$

  is the coefficient of $h^{r-1}$ in $\displaystyle \left(\frac{h}{1-e^{-h}}\right)^r \mathsf{Ch}(\mathcal{F})$.