PROBLEM SESSION
DESCENT TECHNIQUES IN MODULAR REPRESENTATION THEORY

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3. LECTURE

Problem 3.1. Prove the *Beck-Chevally condition* for the functor $\underline{\mathcal{C}} :=$ Rep: $(G\text{-Set})^{\mathrm{op}} \to$ Add from G-sets to additive categories, i.e., prove:

For every pullback square in G-Set (as in the commutative square on the left hand side)

$$\begin{tikzcd}
Y' \arrow[r, "\beta'"] \arrow[d, "\alpha'"'] & Y \arrow[d, "\alpha"] \\
X' \arrow[r, "\beta"] & X
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\underline{\mathcal{C}}(X') \arrow[dr, equal, bend right] & \underline{\mathcal{C}}(Y') \arrow[l, "\alpha'_*"'] & \underline{\mathcal{C}}(Y) \arrow[l, "\beta'^*"'] \arrow[dr, equal, bend left] \arrow[dr, phantom, "\nearrow\varepsilon"] & \\
& \underline{\mathcal{C}}(X') \arrow[u, "\alpha'^*"'] \arrow[u, phantom, shift right=4ex, "\nearrow\eta"] & \underline{\mathcal{C}}(X) \arrow[l, "\beta^*"'] \arrow[u, "\alpha^*"] & \underline{\mathcal{C}}(Y) \arrow[l, "\alpha_*"]
\end{tikzcd}$$

the composite of natural transformations

$$\beta^*\alpha_* \xrightarrow{\eta\,\beta^*\alpha_*} \alpha'_*\alpha'^*\beta^*\alpha_* \overset{\beta\alpha'=\alpha\beta'}{=\!=\!=} \alpha'_*\beta'^*\alpha^*\alpha_* \xrightarrow{\alpha'_*\beta'^*\,\varepsilon} \alpha'_*\beta'^*$$

(as obtained by the "pasting" on the right hand side) is an isomorphism of functors $\underline{\mathcal{C}}(X') \to \underline{\mathcal{C}}(Y')$.

Problem 3.2 (Benabou-Roubaud Theorem)**.** Let $\mathcal{G}$ be a site with pullbacks and let $\underline{\mathcal{C}} \colon \mathcal{G}^{\mathrm{op}} \to$ Add be a functor satisfying the Beck-Chevalley property, as above. Let $\alpha \colon U \to X$ be a cover in $\mathcal{G}$, and construct the following pullback square:

$$\begin{tikzcd}
U^{(2)} := U \times_X U \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & U \arrow[d, "\alpha"] \\
U \arrow[r, "\alpha"] & X
\end{tikzcd}$$

Let $\big(W, \gamma \colon \mathrm{pr}_2^*(W) \overset{\sim}{\to} \mathrm{pr}_1^*(W)\big) \in \mathrm{Desc}_{\underline{\mathcal{C}}}(U)$ be an object in the descent category associated with the covering. Applying the Beck-Chevalley property to the above pullback, we obtain a morphism

$$\gamma' \colon W \longrightarrow (\mathrm{pr}_2)_*\mathrm{pr}_1^*(W) \overset{BC^{-1}}{\cong} \alpha^*\alpha_*(W) =: L(W)$$

where the first map is obtained from γ by adjunction. Prove the following:

(1) The data $B(W,\gamma) := (W,\gamma')$ is a comodule over the comonad $L = (\alpha^*\alpha_*, \Delta, \varepsilon)$ defined by the adjunction α^*, α_*.
(2) The assignment $(W,\gamma) \mapsto (W,\gamma')$ defines an equivalence of categories B making the following triangle commute

$$\begin{tikzcd}
\mathrm{Desc}_{\underline{\mathcal{C}}}(U) \arrow[rr, "B", "\sim"'] & & L\text{-Comod}_{\underline{\mathcal{C}}(U)} \\
& \underline{\mathcal{C}}(X) \arrow[ul, "Q"] \arrow[ur, "E"'] &
\end{tikzcd}$$

up to an isomorphism of functors (here Q and E are the always-defined canonical functors).
(3) Conclude that $\underline{\mathcal{C}}$ satisfies descent with respect to the covering $\alpha\colon U \to X$ if and only if the adjunction α^*, α_* is comonadic.

Problem 3.3. When $[G\colon H]$ is prime to $p = \mathrm{char}(k)$, show that the unit of the adjunction $\mathrm{Res}\colon \mathcal{C}(G) \leftrightarrows \mathcal{C}(H) : \mathrm{Ind}$ is a naturally split mono.

Problem 3.4. Given an adjunction $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : G$ of abelian (respectively, Frobenius abelian) categories such that the unit is naturally split, as in the previous exercise, when is the unit of the induced derived (resp. stable) adjunction also split?