3

Introduction to Linear Programming

Linear programming (LP) is a tool for solving optimization problems. In 1947, George Dantzig developed an efficient method, the simplex algorithm, for solving linear programming problems (also called LP). Since the development of the simplex algorithm, LP has been used to solve optimization problems in industries as diverse as banking, education, forestry, petroleum, and trucking. In a survey of Fortune 500 firms, 85% of the respondents said they had used linear programming. As a measure of the importance of linear programming in operations research, approximately 70% of this book will be devoted to linear programming and related optimization techniques.

In Section 3.1, we begin our study of linear programming by describing the general characteristics shared by all linear programming problems. In Sections 3.2 and 3.3, we learn how to solve graphically those linear programming problems that involve only two variables. Solving these simple LPs will give us useful insights for solving more complex LPs. The remainder of the chapter explains how to formulate linear programming models of real-life situations.

3.1 What Is a Linear Programming Problem?

In this section, we introduce linear programming and define important terms that are used to describe linear programming problems.

EXAMPLE 1 Giapetto's Woodcarving

Giapetto's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains. A soldier sells for $27 and uses $10 worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by $14. A train sells for $21 and uses $9 worth of raw materials. Each train built increases Giapetto's variable labor and overhead costs by $10. The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing. A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing and 1 hour of carpentry labor. Each week, Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for trains is unlimited, but at most 40 soldiers are bought each week. Giapetto wants to maximize weekly profit (revenues − costs). Formulate a mathematical model of Giapetto's situation that can be used to maximize Giapetto's weekly profit.

Solution In developing the Giapetto model, we explore characteristics shared by all linear programming problems.

Decision Variables We begin by defining the relevant **decision variables.** In any linear programming model, the decision variables should completely describe the decisions to be made (in this case, by Giapetto). Clearly, Giapetto must decide how many soldiers and trains should be manufactured each week. With this in mind, we define

$$x_1 = \text{number of soldiers produced each week}$$
$$x_2 = \text{number of trains produced each week}$$

Objective Function In any linear programming problem, the decision maker wants to maximize (usually revenue or profit) or minimize (usually costs) some function of the decision variables. The function to be maximized or minimized is called the **objective function.** For the Giapetto problem, we note that *fixed costs* (such as rent and insurance) do not depend on the values of x_1 and x_2. Thus, Giapetto can concentrate on maximizing (weekly revenues) − (raw material purchase costs) − (other variable costs).

Giapetto's weekly revenues and costs can be expressed in terms of the decision variables x_1 and x_2. It would be foolish for Giapetto to manufacture more soldiers than can be sold, so we assume that all toys produced will be sold. Then

$$\begin{aligned}\text{Weekly revenues} &= \text{weekly revenues from soldiers}\\ &\quad + \text{weekly revenues from trains}\\ &= \left(\frac{\text{dollars}}{\text{soldier}}\right)\left(\frac{\text{soldiers}}{\text{week}}\right) + \left(\frac{\text{dollars}}{\text{train}}\right)\left(\frac{\text{trains}}{\text{week}}\right)\\ &= 27x_1 + 21x_2\end{aligned}$$

Also,

$$\begin{aligned}\text{Weekly raw material costs} &= 10x_1 + 9x_2\\ \text{Other weekly variable costs} &= 14x_1 + 10x_2\end{aligned}$$

Then Giapetto wants to maximize

$$(27x_1 + 21x_2) - (10x_1 + 9x_2) - (14x_1 + 10x_2) = 3x_1 + 2x_2$$

Another way to see that Giapetto wants to maximize $3x_1 + 2x_2$ is to note that

$$\begin{aligned}\text{Weekly revenues} &= \text{weekly contribution to profit from soldiers}\\ &\quad - \text{weekly nonfixed costs} + \text{weekly contribution to profit from trains}\\ &= \left(\frac{\text{contribution to profit}}{\text{soldier}}\right)\left(\frac{\text{soldiers}}{\text{week}}\right)\\ &\quad + \left(\frac{\text{contribution to profit}}{\text{train}}\right)\left(\frac{\text{trains}}{\text{week}}\right)\end{aligned}$$

Also,

$$\frac{\text{Contribution to profit}}{\text{Soldier}} = 27 - 10 - 14 = 3$$
$$\frac{\text{Contribution to profit}}{\text{Train}} = 21 - 9 - 10 = 2$$

Then, as before, we obtain

$$\text{Weekly revenues} - \text{weekly nonfixed costs} = 3x_1 + 2x_2$$

Thus, Giapetto's objective is to choose x_1 and x_2 to maximize $3x_1 + 2x_2$. We use the variable z to denote the objective function value of any LP. Giapetto's objective function is

$$\text{Maximize } z = 3x_1 + 2x_2 \tag{1}$$

(In the future, we will abbreviate "maximize" by *max* and "minimize" by *min.*) The coefficient of a variable in the objective function is called the **objective function coefficient** of the variable. For example, the objective function coefficient for x_1 is 3, and the objective function coefficient for x_2 is 2. In this example (and in many other problems), the objective function coefficient for each variable is simply the contribution of the variable to the company's profit.

Constraints As x_1 and x_2 increase, Giapetto's objective function grows larger. This means that if Giapetto were free to choose any values for x_1 and x_2, the company could make an arbitrarily large profit by choosing x_1 and x_2 to be very large. Unfortunately, the values of x_1 and x_2 are limited by the following three restrictions (often called **constraints**):

Constraint 1 Each week, no more than 100 hours of finishing time may be used.

Constraint 2 Each week, no more than 80 hours of carpentry time may be used.

Constraint 3 Because of limited demand, at most 40 soldiers should be produced each week.

The amount of raw material available is assumed to be unlimited, so no restrictions have been placed on this.

The next step in formulating a mathematical model of the Giapetto problem is to express Constraints 1–3 in terms of the decision variables x_1 and x_2. To express Constraint 1 in terms of x_1 and x_2, note that

$$\begin{aligned}\frac{\text{Total finishing hrs.}}{\text{Week}} &= \left(\frac{\text{finishing hrs.}}{\text{soldier}}\right)\left(\frac{\text{soldiers made}}{\text{week}}\right) \\ &\quad + \left(\frac{\text{finishing hrs.}}{\text{train}}\right)\left(\frac{\text{trains made}}{\text{week}}\right) \\ &= 2(x_1) + 1(x_2) = 2x_1 + x_2\end{aligned}$$

Now Constraint 1 may be expressed by

$$2x_1 + x_2 \leq 100 \tag{2}$$

Note that the units of each term in (2) are finishing hours per week. *For a constraint to be reasonable, all terms in the constraint must have the same units.* Otherwise one is adding apples and oranges, and the constraint won't have any meaning.

To express Constraint 2 in terms of x_1 and x_2, note that

$$\begin{aligned}\frac{\text{Total carpentry hrs.}}{\text{Week}} &= \left(\frac{\text{carpentry hrs.}}{\text{solider}}\right)\left(\frac{\text{soldiers}}{\text{week}}\right) \\ &\quad + \left(\frac{\text{carpentry hrs.}}{\text{train}}\right)\left(\frac{\text{trains}}{\text{week}}\right)\end{aligned}$$

$$= 1(x_1) + 1(x_2) = x_1 + x_2$$

Then Constraint 2 may be written as

$$x_1 + x_2 \leq 80 \tag{3}$$

Again, note that the units of each term in (3) are the same (in this case, carpentry hours per week).

Finally, we express the fact that at most 40 soldiers per week can be sold by limiting the weekly production of soldiers to at most 40 soldiers. This yields the following constraint:

$$x_1 \leq 40 \tag{4}$$

Thus (2)–(4) express Constraints 1–3 in terms of the decision variables; they are called the *constraints* for the Giapetto linear programming problem. The coefficients of the decision variables in the constraints are called **technological coefficients.** This is because the technological coefficients often reflect the technology used to produce different products. For example, the technological coefficient of x_2 in (3) is 1, indicating that a soldier requires 1 carpentry hour. The number on the right-hand side of each constraint is called the constraint's **right-hand side** (or **rhs**). Often the rhs of a constraint represents the quantity of a resource that is available.

Sign Restrictions To complete the formulation of a linear programming problem, the following question must be answered for each decision variable: Can the decision variable only assume nonnegative values, or is the decision variable allowed to assume both positive and negative values?

If a decision variable x_i can only assume nonnegative values, then we add the **sign restriction** $x_i \geq 0$. If a variable x_i can assume both positive and negative (or zero) values, then we say that x_i is **unrestricted in sign** (often abbreviated **urs**). For the Giapetto problem, it is clear that $x_1 \geq 0$ and $x_2 \geq 0$. In other problems, however, some variables may be urs. For example, if x_i represented a firm's cash balance, then x_i could be considered negative if the firm owed more money than it had on hand. In this case, it would be appropriate to classify x_i as urs. Other uses of urs variables are discussed in Section 4.12.

Combining the sign restrictions $x_1 \geq 0$ and $x_2 \geq 0$ with the objective function (1) and Constraints (2)–(4) yields the following optimization model:

$$\max z = 3x_1 + 2x_2 \qquad \text{(Objective function)} \tag{1}$$

subject to (s.t.)

$$2x_1 + x_2 \leq 100 \qquad \text{(Finishing constraint)} \tag{2}$$
$$x_1 + x_2 \leq 80 \qquad \text{(Carpentry constraint)} \tag{3}$$
$$x_1 \leq 40 \qquad \text{(Constraint on demand for soldiers)} \tag{4}$$
$$x_1 \geq 0 \qquad \text{(Sign restriction)}^{\dagger} \tag{5}$$
$$x_2 \geq 0 \qquad \text{(Sign restriction)} \tag{6}$$

"Subject to" (s.t.) means that the values of the decision variables x_1 and x_2 must satisfy all constraints and all sign restrictions.

Before formally defining a linear programming problem, we define the concepts of linear function and linear inequality.

DEFINITION ■ A function $f(x_1, x_2, \ldots, x_n)$ of $x_1, x_2, \ldots, x_n$ is a **linear function** if and only if for some set of constants $c_1, c_2, \ldots, c_n$, $f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$. ■

For example, $f(x_1, x_2) = 2x_1 + x_2$ is a linear function of x_1 and x_2, but $f(x_1, x_2) = x_1^2x_2$ is not a linear function of x_1 and x_2.

DEFINITION ■ For any linear function $f(x_1, x_2, \ldots, x_n)$ and any number b, the inequalities $f(x_1, x_2, \ldots, x_n) \leq b$ and $f(x_1, x_2, \ldots, x_n) \geq b$ are **linear inequalities.** ■

Thus, $2x_1 + 3x_2 \leq 3$ and $2x_1 + x_2 \geq 3$ are linear inequalities, but $x_1^2x_2 \geq 3$ is not a linear inequality.

†The sign restrictions do constrain the values of the decision variables, but we choose to consider the sign restrictions as being separate from the constraints. The reason for this will become apparent when we study the simplex algorithm in Chapter 4.

DEFINITION ■ A **linear programming problem** (LP) is an optimization problem for which we do the following:

1 We attempt to maximize (or minimize) a *linear* function of the decision variables. The function that is to be maximized or minimized is called the *objective function.*

2 The values of the decision variables must satisfy a set of *constraints.* Each constraint must be a linear equation or linear inequality.

3 A *sign restriction* is associated with each variable. For any variable x_i, the sign restriction specifies that x_i must be either nonnegative ($x_i \geq 0$) or unrestricted in sign (urs). ■

Because Giapetto's objective function is a linear function of x_1 and x_2, and all of Giapetto's constraints are linear inequalities, the Giapetto problem is a linear programming problem. Note that the Giapetto problem is typical of a wide class of linear programming problems in which a decision maker's goal is to maximize profit subject to limited resources.

The Proportionality and Additivity Assumptions

The fact that the objective function for an LP must be a linear function of the decision variables has two implications.

1 The contribution of the objective function from each decision variable is proportional to the value of the decision variable. For example, the contribution to the objective function from making four soldiers ($4 \times 3 = \$12$) is exactly four times the contribution to the objective function from making one soldier (\$3).

2 The contribution to the objective function for any variable is independent of the values of the other decision variables. For example, no matter what the value of x_2, the manufacture of x_1 soldiers will always contribute $3x_1$ dollars to the objective function.

Analogously, the fact that each LP constraint must be a linear inequality or linear equation has two implications.

1 The contribution of each variable to the left-hand side of each constraint is proportional to the value of the variable. For example, it takes exactly three times as many finishing hours ($2 \times 3 = 6$ finishing hours) to manufacture three soldiers as it takes to manufacture one soldier (2 finishing hours).

2 The contribution of a variable to the left-hand side of each constraint is independent of the values of the variable. For example, no matter what the value of x_1, the manufacture of x_2 trains uses x_2 finishing hours and x_2 carpentry hours.

The first implication given in each list is called the **Proportionality Assumption of Linear Programming.** Implication 2 of the first list implies that the value of the objective function is the sum of the contributions from individual variables, and implication 2 of the second list implies that the left-hand side of each constraint is the sum of the contributions from each variable. For this reason, the second implication in each list is called the **Additivity Assumption of Linear Programming.**

For an LP to be an appropriate representation of a real-life situation, the decision variables must satisfy both the Proportionality and Additivity Assumptions. Two other assumptions must also be satisfied before an LP can appropriately represent a real situation: the Divisibility and Certainty Assumptions.

The Divisibility Assumption

The **Divisibility Assumption** requires that each decision variable be allowed to assume fractional values. For example, in the Giapetto problem, the Divisibility Assumption implies that it is acceptable to produce 1.5 soldiers or 1.63 trains. Because Giapetto cannot actually produce a fractional number of trains or soldiers, the Divisibility Assumption is not satisfied in the Giapetto problem. A linear programming problem in which some or all of the variables must be nonnegative integers is called an **integer programming problem.** The solution of integer programming problems is discussed in Chapter 9.

In many situations where divisibility is not present, rounding off each variable in the optimal LP solution to an integer may yield a reasonable solution. Suppose the optimal solution to an LP stated that an auto company should manufacture 150,000.4 compact cars during the current year. In this case, you could tell the auto company to manufacture 150,000 or 150,001 compact cars and be fairly confident that this would reasonably approximate an optimal production plan. On the other hand, if the number of missile sites that the United States should use were a variable in an LP and the optimal LP solution said that 0.4 missile sites should be built, it would make a big difference whether we rounded the number of missile sites down to 0 or up to 1. In this situation, the integer programming methods of Chapter 9 would have to be used, because the number of missile sites is definitely not divisible.

The Certainty Assumption

The **Certainty Assumption** is that each parameter (objective function coefficient, right-hand side, and technological coefficient) is known with certainty. If we were unsure of the exact amount of carpentry and finishing hours required to build a train, the Certainty Assumption would be violated.

Feasible Region and Optimal Solution

Two of the most basic concepts associated with a linear programming problem are feasible region and optimal solution. For defining these concepts, we use the term *point* to mean a specification of the value for each decision variable.

DEFINITION ■ The **feasible region** for an LP is the set of all points that satisfies all the LP's constraints and sign restrictions. ■

For example, in the Giapetto problem, the point ($x_1 = 40$, $x_2 = 20$) is in the feasible region. Note that $x_1 = 40$ and $x_2 = 20$ satisfy the constraints (2)–(4) and the sign restrictions (5)–(6):

Constraint (2), $2x_1 + x_2 \le 100$, is satisfied, because $2(40) + 20 \le 100$.

Constraint (3), $x_1 + x_2 \le 80$, is satisfied, because $40 + 20 \le 80$.

Constraint (4), $x_1 \le 40$, is satisfied, because $40 \le 40$.

Restriction (5), $x_1 \ge 0$, is satisfied, because $40 \ge 0$.

Restriction (6), $x_2 \ge 0$, is satisfied, because $20 \ge 0$.

On the other hand, the point ($x_1 = 15$, $x_2 = 70$) is not in the feasible region, because even though $x_1 = 15$ and $x_2 = 70$ satisfy (2), (4), (5), and (6), they fail to satisfy (3): $15 + 70$ is not less than or equal to 80. Any point that is not in an LP's feasible region is said to be an **infeasible point.** As another example of an infeasible point, consider ($x_1 = 40$, $x_2 = -20$). Although this point satisfies all the constraints and the sign restriction (5), it is infeasible because it fails to satisfy the sign restriction (6), $x_2 \geq 0$. The feasible region for the Giapetto problem is the set of possible production plans that Giapetto must consider in searching for the optimal production plan.

DEFINITION ■ For a maximization problem, an **optimal solution** to an LP is a point in the feasible region with the largest objective function value. Similarly, for a minimization problem, an optimal solution is a point in the feasible region with the smallest objective function value. ■

Most LPs have only one optimal solution. However, some LPs have no optimal solution, and some LPs have an infinite number of solutions (these situations are discussed in Section 3.3). In Section 3.2, we show that the unique optimal solution to the Giapetto problem is ($x_1 = 20$, $x_2 = 60$). This solution yields an objective function value of

$$z = 3x_1 + 2x_2 = 3(20) + 2(60) = \$180$$

When we say that ($x_1 = 20$, $x_2 = 60$) is the optimal solution to the Giapetto problem, we are saying that no point in the feasible region has an objective function value that exceeds 180. Giapetto can maximize profit by building 20 soldiers and 60 trains each week. If Giapetto were to produce 20 soldiers and 60 trains each week, the weekly profit would be \$180 less weekly fixed costs. For example, if Giapetto's only fixed cost were rent of \$100 per week, then weekly profit would be $180 - 100 = \$80$ per week.

PROBLEMS

Group A

1 Farmer Jones must determine how many acres of corn and wheat to plant this year. An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week. An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week. All wheat can be sold at \$4 a bushel, and all corn can be sold at \$3 a bushel. Seven acres of land and 40 hours per week of labor are available. Government regulations require that at least 30 bushels of corn be produced during the current year. Let x_1 = number of acres of corn planted, and x_2 = number of acres of wheat planted. Using these decision variables, formulate an LP whose solution will tell Farmer Jones how to maximize the total revenue from wheat and corn.

2 Answer these questions about Problem 1.

- **a** Is ($x_1 = 2$, $x_2 = 3$) in the feasible region?
- **b** Is ($x_1 = 4$, $x_2 = 3$) in the feasible region?
- **c** Is ($x_1 = 2$, $x_2 = -1$) in the feasible region?
- **d** Is ($x_1 = 3$, $x_2 = 2$) in the feasible region?

3 Using the variables x_1 = number of bushels of corn produced and x_2 = number of bushels of wheat produced, reformulate Farmer Jones's LP.

4 Truckco manufactures two types of trucks: 1 and 2. Each truck must go through the painting shop and assembly shop. If the painting shop were completely devoted to painting Type 1 trucks, then 800 per day could be painted; if the painting shop were completely devoted to painting Type 2 trucks, then 700 per day could be painted. If the assembly shop were completely devoted to assembling truck 1 engines, then 1,500 per day could be assembled; if the assembly shop were completely devoted to assembling truck 2 engines, then 1,200 per day could be assembled. Each Type 1 truck contributes \$300 to profit; each Type 2 truck contributes \$500. Formulate an LP that will maximize Truckco's profit.

Group B

5 Why don't we allow an LP to have $<$ or $>$ constraints?

3.2 The Graphical Solution of Two-Variable Linear Programming Problems

Any LP with only two variables can be solved graphically. We always label the variables x_1 and x_2 and the coordinate axes the x_1 and x_2 axes. Suppose we want to graph the set of points that satisfies

$$2x_1 + 3x_2 \leq 6 \tag{7}$$

The same set of points (x_1, x_2) satisfies

$$3x_2 \leq 6 - 2x_1$$

This last inequality may be rewritten as

$$x_2 \leq \tfrac{1}{3}(6 - 2x_1) = 2 - \tfrac{2}{3}x_1 \tag{8}$$

Because moving downward on the graph decreases x_2 (see Figure 1), the set of points that satisfies (8) and (7) lies on or below the line $x_2 = 2 - \frac{2}{3}x_1$. This set of points is indicated by darker shading in Figure 1. Note, however, that $x_2 = 2 - \frac{2}{3}x_1$, $3x_2 = 6 - 2x_1$, and $2x_1 + 3x_2 = 6$ are all the same line. This means that the set of points satisfying (7) lies on or below the line $2x_1 + 3x_2 = 6$. Similarly, the set of points satisfying $2x_1 + 3x_2 \geq 6$ lies on or above the line $2x_1 + 3x_2 = 6$. (These points are shown by lighter shading in Figure 1.)

Consider a linear inequality constraint of the form $f(x_1, x_2) \geq b$ or $f(x_1, x_2) \leq b$. In general, it can be shown that in two dimensions, the set of points that satisfies a linear inequality includes the points on the line $f(x_1, x_2) = b$, defining the inequality plus all points on one side of the line.

There is an easy way to determine the side of the line for which an inequality such as $f(x_1, x_2) \leq b$ or $f(x_1, x_2) \geq b$ is satisfied. Just choose any point P that does not satisfy the line $f(x_1, x_2) = b$. Determine whether P satisfies the inequality. If it does, then all points on the *same* side as P of $f(x_1, x_2) = b$ will satisfy the inequality. If P does not satisfy the inequality, then all points on the *other* side of $f(x_1, x_2) = b$, which does not contain P, will satisfy the inequality. For example, to determine whether $2x_1 + 3x_2 \geq 6$ is satisfied by points above or below the line $2x_1 + 3x_2 = 6$, we note that (0, 0) does not satisfy $2x_1 + 3x_2 \geq 6$. Because (0, 0) is *below* the line $2x_1 + 3x_2 = 6$, the set of points satisfying $2x_1 + 3x_2 \geq 6$ includes the line $2x_1 + 3x_2 = 6$ and the points *above* the line $2x_1 + 3x_2 = 6$. This agrees with Figure 1.

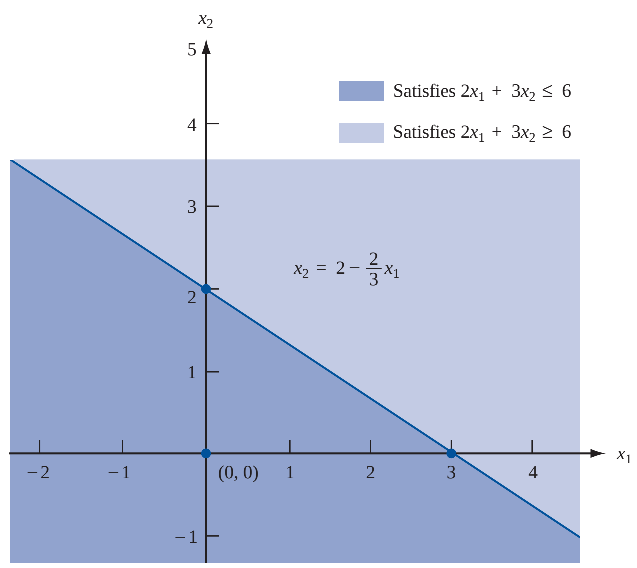


FIGURE 1
Graphing a Linear Inequality

Finding the Feasible Solution

We illustrate how to solve two-variable LPs graphically by solving the Giapetto problem. To begin, we graphically determine the feasible region for Giapetto's problem. The feasible region for the Giapetto problem is the set of all points (x_1, x_2) satisfying

$$2x_1 + x_2 \le 100 \qquad \text{(Constraints)} \tag{2}$$
$$x_1 + x_2 \le 80 \tag{3}$$
$$x_1 \le 40 \tag{4}$$
$$x_1 \ge 0 \qquad \text{(Sign restrictions)} \tag{5}$$
$$x_2 \ge 0 \tag{6}$$

For a point (x_1, x_2) to be in the feasible region, (x_1, x_2) must satisfy *all* the inequalities (2)–(6). Note that the only points satisfying (5) and (6) lie in the first quadrant of the x_1–x_2 plane. This is indicated in Figure 2 by the arrows pointing to the right from the x_2 axis and upward from the x_1 axis. Thus, any point that is outside the first quadrant cannot be in the feasible region. This means that the feasible region will be the set of points in the first quadrant that satisfies (2)–(4).

Our method for determining the set of points that satisfies a linear inequality will also identify those that meet (2)–(4). From Figure 2, we see that (2) is satisfied by all points below or on the line *AB* (*AB* is the line $2x_1 + x_2 = 100$). Inequality (3) is satisfied by all points on or below the line *CD* (*CD* is the line $x_1 + x_2 = 80$). Finally, (4) is satisfied by all points on or to the left of line *EF* (*EF* is the line $x_1 = 40$). The side of a line that satisfies an inequality is indicated by the direction of the arrows in Figure 2.

From Figure 2, we see that the set of points in the first quadrant that satisfies (2), (3), and (4) is bounded by the five-sided polygon *DGFEH*. Any point on this polygon or in its interior is in the feasible region. Any other point fails to satisfy at least one of the inequalities (2)–(6). For example, the point (40, 30) lies outside *DGFEH* because it is above the line segment *AB*. Thus (40, 30) is infeasible, because it fails to satisfy (2).

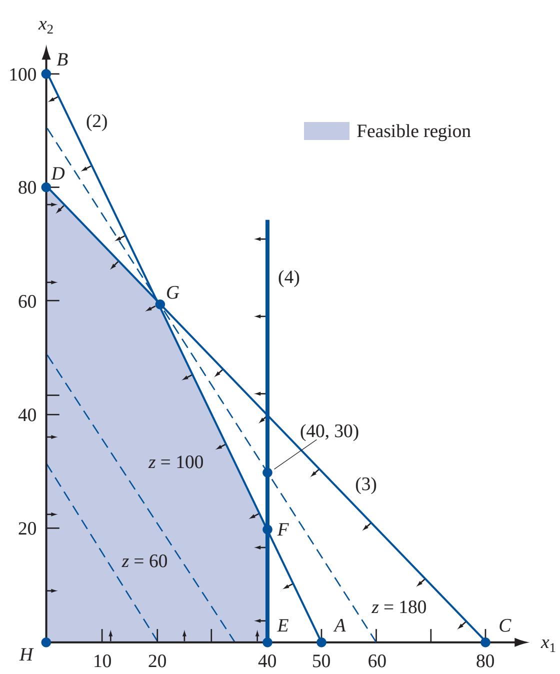


FIGURE 2
Graphical Solution of Giapetto Problem

An easy way to find the feasible region is to determine the set of infeasible points. Note that all points above line *AB* in Figure 2 are infeasible, because they fail to satisfy (2). Similarly, all points above *CD* are infeasible, because they fail to satisfy (3). Also, all points to the right of the vertical line *EF* are infeasible, because they fail to satisfy (4). After these points are eliminated from consideration, we are left with the feasible region (*DGFEH*).

Finding the Optimal Solution

Having identified the feasible region for the Giapetto problem, we now search for the optimal solution, which will be the point in the feasible region with the largest value of $z = 3x_1 + 2x_2$. To find the optimal solution, we need to graph a line on which all points have the same z-value. In a max problem, such a line is called an **isoprofit line** (in a min problem, an **isocost line**). To draw an isoprofit line, choose any point in the feasible region and calculate its z-value. Let us choose (20, 0). For (20, 0), $z = 3(20) + 2(0) = 60$. Thus, (20, 0) lies on the isoprofit line $z = 3x_1 + 2x_2 = 60$. Rewriting $3x_1 + 2x_2 = 60$ as $x_2 = 30 - \frac{3}{2}x_1$, we see that the isoprofit line $3x_1 + 2x_2 = 60$ has a slope of $-\frac{3}{2}$. Because all isoprofit lines are of the form $3x_1 + 2x_2 =$ constant, all isoprofit lines have the same slope. *This means that once we have drawn one isoprofit line, we can find all other isoprofit lines by moving parallel to the isoprofit line we have drawn.*

It is now clear how to find the optimal solution to a two-variable LP. After you have drawn a single isoprofit line, generate other isoprofit lines by moving parallel to the drawn isoprofit line in a direction that increases z (for a max problem). After a point, the isoprofit lines will no longer intersect the feasible region. The last isoprofit line intersecting (touching) the feasible region defines the largest z-value of any point in the feasible region and indicates the optimal solution to the LP. In our problem, the objective function $z = 3x_1 + 2x_2$ will increase if we move in a direction for which both x_1 and x_2 increase. Thus, we construct additional isoprofit lines by moving parallel to $3x_1 + 2x_2 = 60$ in a northeast direction (upward and to the right). From Figure 2, we see that the isoprofit line passing through point *G* is the last isoprofit line to intersect the feasible region. Thus, *G* is the point in the feasible region with the largest z-value and is therefore the optimal solution to the Giapetto problem. Note that point *G* is where the lines $2x_1 + x_2 = 100$ and $x_1 + x_2 = 80$ intersect. Solving these two equations simultaneously, we find that ($x_1 = 20$, $x_2 = 60$) is the optimal solution to the Giapetto problem. The optimal value of z may be found by substituting these values of x_1 and x_2 into the objective function. Thus, the optimal value of z is $z = 3(20) + 2(60) = 180$.

Binding and Nonbinding Constraints

Once the optimal solution to an LP has been found, it is useful (see Chapters 5 and 6) to classify each constraint as being a binding constraint or a nonbinding constraint.

DEFINITION ■ A constraint is **binding** if the left-hand side and the right-hand side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint. ■

Thus, (2) and (3) are binding constraints.

DEFINITION ■ A constraint is **nonbinding** if the left-hand side and the right-hand side of the constraint are unequal when the optimal values of the decision variables are substituted into the constraint. ■

Because $x_1 = 20$ is less than 40, (4) is a nonbinding constraint.

Convex Sets, Extreme Points, and LP

The feasible region for the Giapetto problem is an example of a convex set.

DEFINITION ■ A set of points *S* is a **convex set** if the line segment joining any pair of points in *S* is wholly contained in *S*. ■

Figure 3 gives four illustrations of this definition. In Figures 3a and 3b, each line segment joining two points in *S* contains only points in *S*. Thus, in both these figures, *S* is convex. In Figures 3c and 3d, *S* is not convex. In each figure, points *A* and *B* are in *S*, but there are points on the line segment *AB* that are not contained in *S*. In our study of linear programming, a certain type of point in a convex set (called an *extreme point*) is of great interest.

DEFINITION ■ For any convex set *S*, a point *P* in *S* is an **extreme point** if each line segment that lies completely in *S* and contains the point *P* has *P* as an endpoint of the line segment. ■

For example, in Figure 3a, each point on the circumference of the circle is an extreme point of the circle. In Figure 3b, points *A*, *B*, *C*, and *D* are extreme points of *S*. Although point *E* is on the boundary of *S* in Figure 3b, *E* is not an extreme point of *S*. This is because *E* lies on the line segment *AB* (*AB* lies completely in *S*), and *E* is not an endpoint of the line segment *AB*. Extreme points are sometimes called **corner points,** because if the set *S* is a polygon, the extreme points of *S* will be the vertices, or corners, of the polygon.

The feasible region for the Giapetto problem is a convex set. This is no accident: It can be shown that the feasible region for any LP will be a convex set. From Figure 2, we see that the extreme points of the feasible region are simply points *D*, *F*, *E*, *G*, and *H*. It can be shown that the feasible region for any LP has only a finite number of extreme points. Also note that the optimal solution to the Giapetto problem (point *G*) is an extreme point of the feasible region. It can be shown that *any LP that has an optimal solution has an extreme point that is optimal.* This result is very important, because it reduces the set of points that yield an optimal solution from the entire feasible region (which generally contains an *infinite* number of points) to the set of extreme points (a *finite* set).

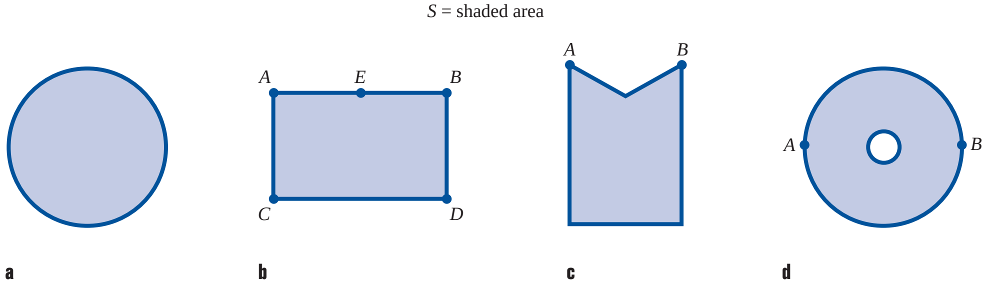


FIGURE 3
Convex and Nonconvex Sets

For the Giapetto problem, it is easy to see why the optimal solution must be an extreme point of the feasible region. We note that z increases as we move isoprofit lines in a northeast direction, so the largest z-value in the feasible region must occur at some point P that has no points in the feasible region northeast of P. This means that the optimal solution must lie somewhere on the boundary of the feasible region *DGFEH*. The LP must have an extreme point that is optimal, because for any line segment on the boundary of the feasible region, the largest z-value on that line segment must be assumed at one of the endpoints of the line segment.

To see this, look at the line segment *FG* in Figure 2. *FG* is part of the line $2x_1 + x_2 = 100$ and has a slope of -2. If we move along *FG* and decrease x_1 by 1, then x_2 will increase by 2, and the value of z changes as follows: $3x_1$ goes down by $3(1) = 3$, and $2x_2$ goes up by $2(2) = 4$. Thus, in total, z increases by $4 - 3 = 1$. This means that moving along *FG* in a direction of decreasing x_1 increases z. Thus, the value of z at point G must exceed the value of z at any other point on the line segment *FG*.

A similar argument shows that for any objective function, the maximum value of z on a given line segment must occur at an endpoint of the line segment. Therefore, for any LP, the largest z-value in the feasible region must be attained at an endpoint of one of the line segments forming the boundary of the feasible region. In short, one of the extreme points of the feasible region must be optimal. (To test your understanding, show that if Giapetto's objective function were $z = 6x_1 + x_2$, point F would be optimal, whereas if Giapetto's objective function were $z = x_1 + 6x_2$, point D would be optimal.)

Our proof that an LP always has an optimal extreme point depended heavily on the fact that both the objective function and the constraints were linear functions. In Chapter 11, we show that for an optimization problem in which the objective function or some of the constraints are not linear, the optimal solution to the optimization problem may not occur at an extreme point.

The Graphical Solution of Minimization Problems

EXAMPLE 2 Dorian Auto

Dorian Auto manufactures luxury cars and trucks. The company believes that its most likely customers are high-income women and men. To reach these groups, Dorian Auto has embarked on an ambitious TV advertising campaign and has decided to purchase 1-minute commercial spots on two types of programs: comedy shows and football games. Each comedy commercial is seen by 7 million high-income women and 2 million high-income men. Each football commercial is seen by 2 million high-income women and 12 million high-income men. A 1-minute comedy ad costs \$50,000, and a 1-minute football ad costs \$100,000. Dorian would like the commercials to be seen by at least 28 million high-income women and 24 million high-income men. Use linear programming to determine how Dorian Auto can meet its advertising requirements at minimum cost.

Solution Dorian must decide how many comedy and football ads should be purchased, so the decision variables are

$$x_1 = \text{number of 1-minute comedy ads purchased}$$
$$x_2 = \text{number of 1-minute football ads purchased}$$

Then Dorian wants to minimize total advertising cost (in thousands of dollars).

$$\begin{aligned}\text{Total advertising cost} &= \text{cost of comedy ads} + \text{cost of football ads}\\ &= \left(\frac{\text{cost}}{\text{comedy ad}}\right)\left(\begin{matrix}\text{total}\\ \text{comedy ads}\end{matrix}\right) + \left(\frac{\text{cost}}{\text{football ad}}\right)\left(\begin{matrix}\text{total}\\ \text{football ads}\end{matrix}\right)\\ &= 50x_1 + 100x_2\end{aligned}$$

Thus, Dorian's objective function is

$$\min z = 50x_1 + 100x_2 \tag{9}$$

Dorian faces the following constraints:

Constraint 1 Commercials must reach at least 28 million high-income women.

Constraint 2 Commercials must reach at least 24 million high-income men.

To express Constraints 1 and 2 in terms of x_1 and x_2, let HIW stand for high-income women viewers and HIM stand for high-income men viewers (in millions).

$$\begin{aligned}\text{HIW} &= \left(\frac{\text{HIW}}{\text{comedy ad}}\right)\left(\begin{matrix}\text{total}\\ \text{comedy ads}\end{matrix}\right) + \left(\frac{\text{HIW}}{\text{football ad}}\right)\left(\begin{matrix}\text{total}\\ \text{football ads}\end{matrix}\right)\\ &= 7x_1 + 2x_2\\ \text{HIM} &= \left(\frac{\text{HIM}}{\text{comedy ad}}\right)\left(\begin{matrix}\text{total}\\ \text{comedy ads}\end{matrix}\right) + \left(\frac{\text{HIM}}{\text{football ad}}\right)\left(\begin{matrix}\text{total}\\ \text{football ads}\end{matrix}\right)\\ &= 2x_1 + 12x_2\end{aligned}$$

Constraint 1 may now be expressed as

$$7x_1 + 2x_2 \geq 28 \tag{10}$$

and Constraint 2 may be expressed as

$$2x_1 + 12x_2 \geq 24 \tag{11}$$

The sign restrictions $x_1 \geq 0$ and $x_2 \geq 0$ are necessary, so the Dorian LP is given by:

$$\begin{aligned}\min z = 50x_1 &+ 100x_2 & \\ \text{s.t.} \quad 7x_1 + \ \ 2x_2 &\geq 28 & \text{(HIW)}\\ 2x_1 + 12x_2 &\geq 24 & \text{(HIM)}\\ x_1, x_2 &\geq 0 &\end{aligned}$$

This problem is typical of a wide range of LP applications in which a decision maker wants to minimize the cost of meeting a certain set of requirements. To solve this LP graphically, we begin by graphing the feasible region (Figure 4). Note that (10) is satisfied by points on or above the line AB (AB is part of the line $7x_1 + 2x_2 = 28$) and that

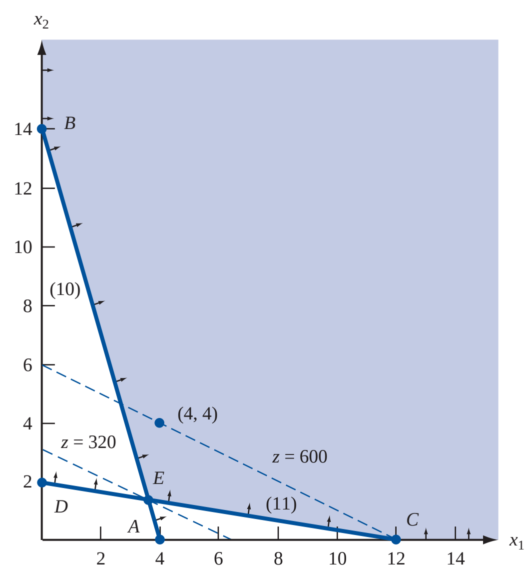


FIGURE 4
Graphical Solution of Dorian Problem

(11) is satisfied by the points on or above the line CD (CD is part of the line $2x_1 + 12x_2 = 24$). From Figure 4, we see that the only first-quadrant points satisfying both (10) and (11) are the points in the shaded region bounded by the x_1 axis, CEB, and the x_2 axis.

Like the Giapetto problem, the Dorian problem has a convex feasible region, but the feasible region for Dorian, unlike Giapetto's, contains points for which the value of at least one variable can assume arbitrarily large values. Such a feasible region is called an **unbounded feasible region.**

Because Dorian wants to minimize total advertising cost, the optimal solution to the problem is the point in the feasible region with the *smallest* z-value. To find the optimal solution, we need to draw an *isocost line* that intersects the feasible region. An isocost line is any line on which all points have the same z-value (or same cost). We arbitrarily choose the isocost line passing through the point ($x_1 = 4$, $x_2 = 4$). For this point, $z = 50(4) + 100(4) = 600$, and we graph the isocost line $z = 50x_1 + 100x_2 = 600$.

We consider lines parallel to the isocost line $50x_1 + 100x_2 = 600$ in the direction of decreasing z (southwest). The last point in the feasible region that intersects an isocost line will be the point in the feasible region having the *smallest* z-value. From Figure 4, we see that point E has the smallest z-value of any point in the feasible region; this is the optimal solution to the Dorian problem. Note that point E is where the lines $7x_1 + 2x_2 = 28$ and $2x_1 + 12x_2 = 24$ intersect. Simultaneously solving these equations yields the optimal solution ($x_1 = 3.6$, $x_2 = 1.4$). The optimal z-value can then be found by substituting these values of x_1 and x_2 into the objective function. Thus, the optimal z-value is $z = 50(3.6) + 100(1.4) = 320 = \$320,000$. Because at point E both the HIW and HIM constraints are satisfied with equality, both constraints are binding.

Does the Dorian model meet the four assumptions of linear programming outlined in Section 3.1?

For the Proportionality Assumption to be valid, each extra comedy commercial must add exactly 7 million HIW and 2 million HIM. This contradicts empirical evidence, which indicates that after a certain point advertising yields diminishing returns. After, say, 500 auto commercials have been aired, most people have probably seen one, so it does little good to air more commercials. Thus, the Proportionality Assumption is violated.

We used the Additivity Assumption to justify writing (total HIW viewers) = (HIW viewers from comedy ads) + (HIW viewers from football ads). In reality, many of the same people will see a Dorian comedy commercial and a Dorian football commercial. We are double-counting such people, and this creates an inaccurate picture of the total number of people seeing Dorian commercials. The fact that the same person may see more than one type of commercial means that the effectiveness of, say, a comedy commercial depends on the number of football commercials. This violates the Additivity Assumption.

If only 1-minute commercials are available, then it is unreasonable to say that Dorian should buy 3.6 comedy commercials and 1.4 football commercials, so the Divisibility Assumption is violated, and the Dorian problem should be considered an integer programming problem. In Section 9.3, we show that if the Dorian problem is solved as an integer programming problem, then the minimum cost is attained by choosing ($x_1 = 6$, $x_2 = 1$) or ($x_1 = 4$, $x_2 = 2$). For either solution, the minimum cost is \$400,000. This is 25% higher than the cost obtained from the optimal LP solution.

Because there is no way to know with certainty how many viewers are added by each type of commercial, the Certainty Assumption is also violated. Thus, all the assumptions of linear programming seem to be violated by the Dorian Auto problem. Despite these drawbacks, analysts have used similar models to help companies determine their optimal media mix.[†]

[†]Lilien and Kotler (1983).

PROBLEMS

Group A

1 Graphically solve Problem 1 of Section 3.1.

2 Graphically solve Problem 4 of Section 3.1.

3 Leary Chemical manufactures three chemicals: A, B, and C. These chemicals are produced via two production processes: 1 and 2. Running process 1 for an hour costs \$4 and yields 3 units of A, 1 of B, and 1 of C. Running process 2 for an hour costs \$1 and produces 1 unit of A and 1 of B. To meet customer demands, at least 10 units of A, 5 of B, and 3 of C must be produced daily. Graphically determine a daily production plan that minimizes the cost of meeting Leary Chemical's daily demands.

4 For each of the following, determine the direction in which the objective function increases:

a $z = 4x_1 - x_2$

b $z = -x_1 + 2x_2$

c $z = -x_1 - 3x_2$

5 Furnco manufactures desks and chairs. Each desk uses 4 units of wood, and each chair uses 3. A desk contributes \$40 to profit, and a chair contributes \$25. Marketing restrictions require that the number of chairs produced be at least twice the number of desks produced. If 20 units of wood are available, formulate an LP to maximize Furnco's profit. Then graphically solve the LP.

6 Farmer Jane owns 45 acres of land. She is going to plant each with wheat or corn. Each acre planted with wheat yields \$200 profit; each with corn yields \$300 profit. The labor and fertilizer used for each acre are given in Table 1. One hundred workers and 120 tons of fertilizer are available. Use linear programming to determine how Jane can maximize profits from her land.

TABLE 1

	Wheat	Corn
Labor	3 workers	2 workers
Fertilizer	2 tons	4 tons

3.3 Special Cases

The Giapetto and Dorian problems each had a unique optimal solution. In this section, we encounter three types of LPs that do not have unique optimal solutions.

1 Some LPs have an infinite number of optimal solutions (*alternative* or *multiple optimal solutions*).

2 Some LPs have no feasible solutions (*infeasible* LPs).

3 Some LPs are *unbounded:* There are points in the feasible region with arbitrarily large (in a max problem) z-values.

Alternative or Multiple Optimal Solutions

EXAMPLE 3 Alternative Optimal Solutions

An auto company manufactures cars and trucks. Each vehicle must be processed in the paint shop and body assembly shop. If the paint shop were only painting trucks, then 40 per day could be painted. If the paint shop were only painting cars, then 60 per day could be painted. If the body shop were only producing cars, then it could process 50 per day. If the body shop were only producing trucks, then it could process 50 per day. Each truck contributes \$300 to profit, and each car contributes \$200 to profit. Use linear programming to determine a daily production schedule that will maximize the company's profits.

Solution The company must decide how many cars and trucks should be produced daily. This leads us to define the following decision variables:

$$x_1 = \text{number of trucks produced daily}$$
$$x_2 = \text{number of cars produced daily}$$

The company's daily profit (in hundreds of dollars) is $3x_1 + 2x_2$, so the company's objective function may be written as

$$\max z = 3x_1 + 2x_2 \tag{12}$$

The company's two constraints are the following:

Constraint 1 The fraction of the day during which the paint shop is busy is less than or equal to 1.

Constraint 2 The fraction of the day during which the body shop is busy is less than or equal to 1.

We have

$$\begin{aligned}
\text{Fraction of day paint shop works on trucks} &= \left(\frac{\text{fraction of day}}{\text{truck}}\right)\left(\frac{\text{trucks}}{\text{day}}\right) \\
&= \tfrac{1}{40}\, x_1 \\
\text{Fraction of day paint shop works on cars} &= \tfrac{1}{60}\, x_2 \\
\text{Fraction of day body shop works on trucks} &= \tfrac{1}{50}\, x_1 \\
\text{Fraction of day body shop works on cars} &= \tfrac{1}{50}\, x_2
\end{aligned}$$

Thus, Constraint 1 may be expressed by

$$\frac{1}{40}x_1 + \frac{1}{60}x_2 \le 1 \qquad \text{(Paint shop constraint)} \tag{13}$$

and Constraint 2 may be expressed by

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \le 1 \qquad \text{(Body shop constraint)} \tag{14}$$

Because $x_1 \ge 0$ and $x_2 \ge 0$ must hold, the relevant LP is

$$\begin{aligned}
&\max z = 3x_1 + 2x_2 && (12)\\
&\text{s.t.} \quad \frac{1}{40}x_1 + \frac{1}{60}x_2 \le 1 && (13)\\
&\qquad\ \ \frac{1}{50}x_1 + \frac{1}{50}x_2 \le 1 && (14)\\
&\qquad\qquad x_1, x_2 \ge 0
\end{aligned}$$

The feasible region for this LP is the shaded region in Figure 5 bounded by *AEDF*.†

For our isoprofit line, we choose the line passing through the point (20, 0). Because (20, 0) has a z-value of $3(20) + 2(0) = 60$, this yields the isoprofit line $z = 3x_1 + 2x_2 = 60$. Examining lines parallel to this isoprofit line in the direction of increasing z (northeast), we find that the last "point" in the feasible region to intersect an isoprofit line is the *entire* line segment *AE*. This means that any point on the line segment *AE* is optimal. We can use any point on *AE* to determine the optimal z-value. For example, point *A*, (40, 0), gives $z = 3(40) = 120$.

In summary, the auto company's LP has an infinite number of optimal solutions, or *multiple* or *alternative optimal solutions*. This is indicated by the fact that as an isoprofit

†Constraint (13) is satisfied by all points on or below *AB* (*AB* is $\frac{1}{40}x_1 + \frac{1}{60}x_2 = 1$), and (14) is satisfied by all points on or below *CD* (*CD* is $\frac{1}{50}x_1 + \frac{1}{50}x_2 = 1$).

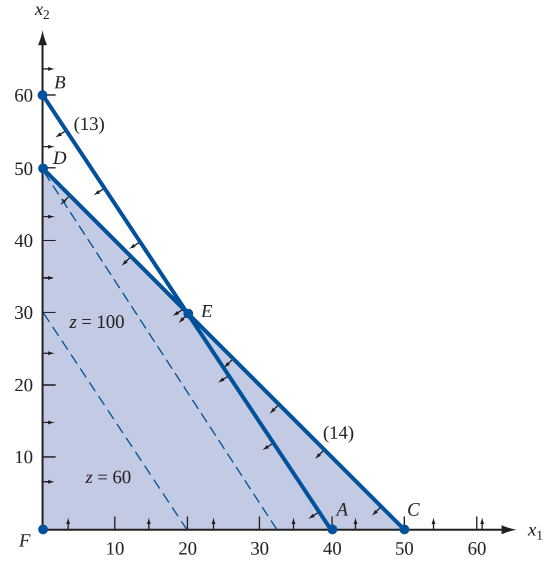


FIGURE 5
Graphical Solution of Example 3

line leaves the feasible region, it will intersect an entire line segment corresponding to the binding constraint (in this case, *AE*).

From our current example, it seems reasonable (and can be shown to be true) that if two points (*A* and *E* here) are optimal, then *any* point on the line segment joining these two points will also be optimal.

If an alternative optimum occurs, then the decision maker can use a secondary criterion to choose between optimal solutions. The auto company's managers might prefer point *A* because it would simplify their business (and still allow them to maximize profits) by allowing them to produce only one type of product (trucks).

The technique of **goal programming** (see Section 4.14) is often used to choose among alternative optimal solutions.

Infeasible LP

It is possible for an LP's feasible region to be empty (contain no points), resulting in an *infeasible* LP. Because the optimal solution to an LP is the best point in the feasible region, an infeasible LP has no optimal solution.

EXAMPLE 4 Infeasible LP

Suppose that auto dealers require that the auto company in Example 3 produce at least 30 trucks and 20 cars. Find the optimal solution to the new LP.

Solution After adding the constraints $x_1 \geq 30$ and $x_2 \geq 20$ to the LP of Example 3, we obtain the following LP:

$$\max z = 3x_1 + 2x_2$$

$$\text{s.t.} \quad \frac{1}{40}x_1 + \frac{1}{60}x_2 \leq 1 \qquad (15)$$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \leq 1 \qquad (16)$$

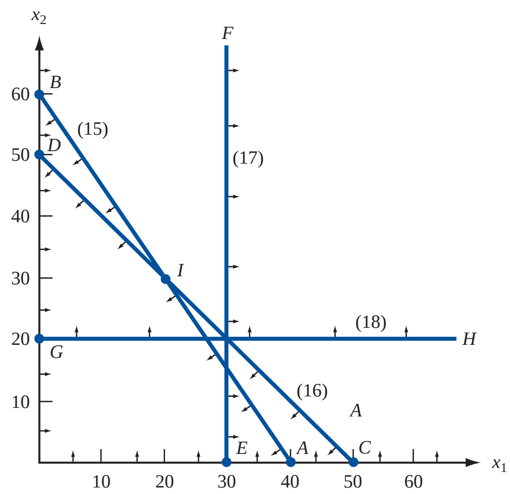


FIGURE 6
An Empty Feasible Region (Infeasible LP)

$$x_1 \geq 30 \qquad \textbf{(17)}$$
$$x_2 \geq 20 \qquad \textbf{(18)}$$
$$x_1, x_2 \geq 0$$

The graph of the feasible region for this LP is Figure 6.

Constraint (15) is satisfied by all points on or below *AB* (*AB* is $\frac{1}{40}x_1 + \frac{1}{60}x_2 = 1$).
Constraint (16) is satisfied by all points on or below *CD* (*CD* is $\frac{1}{50}x_1 + \frac{1}{50}x_2 = 1$).
Constraint (17) is satisfied by all points on or to the right of *EF* (*EF* is $x_1 = 30$).
Constraint (18) is satisfied by all points on or above *GH* (*GH* is $x_2 = 20$).

From Figure 6 it is clear that no point satisfies all of (15)–(18). This means that Example 4 has an empty feasible region and is an infeasible LP.

In Example 4, the LP is infeasible because producing 30 trucks and 20 cars requires more paint shop time than is available.

Unbounded LP

Our next special LP is an *unbounded* LP. For a max problem, an unbounded LP occurs if it is possible to find points in the feasible region with arbitrarily large *z*-values, which corresponds to a decision maker earning arbitrarily large revenues or profits. This would indicate that an unbounded optimal solution should not occur in a correctly formulated LP. Thus, if the reader ever solves an LP on the computer and finds that the LP is unbounded, then an error has probably been made in formulating the LP or in inputting the LP into the computer.

For a minimization problem, an LP is unbounded if there are points in the feasible region with arbitrarily small *z*-values. When graphically solving an LP, we can spot an unbounded LP as follows: A max problem is unbounded if, when we move parallel to our original isoprofit line in the direction of increasing *z*, we never entirely leave the feasible region. A minimization problem is unbounded if we never leave the feasible region when moving in the direction of decreasing *z*.

EXAMPLE 5 **Unbounded LP**

Graphically solve the following LP:

$$\begin{aligned} \max z &= 2x_1 - x_2 \\ \text{s.t.} \quad x_1 - x_2 &\le 1 \qquad (19) \\ 2x_1 + x_2 &\ge 6 \qquad (20) \\ x_1, x_2 &\ge 0 \end{aligned}$$

Solution From Figure 7, we see that (19) is satisfied by all points on or above *AB* (*AB* is the line $x_1 - x_2 = 1$). Also, (20) is satisfied by all points on or above *CD* (*CD* is $2x_1 + x_2 = 6$). Thus, the feasible region for Example 5 is the (shaded) unbounded region in Figure 7, which is bounded only by the x_2 axis, line segment *DE*, and the part of line *AB* beginning at *E*. To find the optimal solution, we draw the isoprofit line passing through (2, 0). This isoprofit line has $z = 2x_1 - x_2 = 2(2) - 0 = 4$. The direction of increasing z is to the southeast (this makes x_1 larger and x_2 smaller). Moving parallel to $z = 2x_1 - x_2$ in a southeast direction, we see that any isoprofit line we draw will intersect the feasible region. (This is because any isoprofit line is steeper than the line $x_1 - x_2 = 1$.)

Thus, there are points in the feasible region that have arbitrarily large z-values. For example, if we wanted to find a point in the feasible region that had $z \ge 1{,}000{,}000$, we could choose any point in the feasible region that is southeast of the isoprofit line $z = 1{,}000{,}000$.

From the discussion in the last two sections, we see that every LP with two variables must fall into one of the following four cases:

Case 1 The LP has a unique optimal solution.

Case 2 The LP has alternative or multiple optimal solutions: Two or more extreme points are optimal, and the LP will have an infinite number of optimal solutions.

Case 3 The LP is infeasible: The feasible region contains no points.

Case 4 The LP is unbounded: There are points in the feasible region with arbitrarily large z-values (max problem) or arbitrarily small z-values (min problem).

In Chapter 4, we show that every LP (not just LPs with two variables) must fall into one of Cases 1–4.

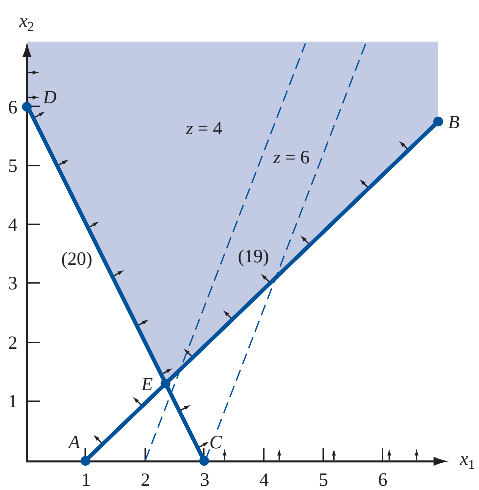


FIGURE 7
An Unbounded LP

In the rest of this chapter, we lead the reader through the formulation of several more complicated linear programming models. The most important step in formulating an LP model is the proper choice of decision variables. If the decision variables have been properly chosen, the objective function and constraints should follow without much difficulty. Trouble in determining an LP's objective function and constraints is usually the result of an incorrect choice of decision variables.

PROBLEMS

Group A

Identify which of Cases 1–4 apply to each of the following LPs:

1
$$\begin{aligned}\max z &= x_1 + x_2\\ \text{s.t.}\quad x_1 + x_2 &\le 4\\ x_1 - x_2 &\ge 5\\ x_1, x_2 &\ge 0\end{aligned}$$

2
$$\begin{aligned}\max z &= 4x_1 + x_2\\ \text{s.t.}\quad 8x_1 + 2x_2 &\le 16\\ 5x_1 + 2x_2 &\le 12\\ x_1, x_2 &\ge 0\end{aligned}$$

3
$$\begin{aligned}\max z &= -x_1 + 3x_2\\ \text{s.t.}\quad x_1 - x_2 &\le 4\\ x_1 + 2x_2 &\ge 4\\ x_1, x_2 &\ge 0\end{aligned}$$

4
$$\begin{aligned}\max z &= 3x_1 + x_2\\ \text{s.t.}\quad 2x_1 + x_2 &\le 6\\ x_1 + 3x_2 &\le 9\\ x_1, x_2 &\ge 0\end{aligned}$$

5 True or false: For an LP to be unbounded, the LP's feasible region must be unbounded.

6 True or false: Every LP with an unbounded feasible region has an unbounded optimal solution.

7 If an LP's feasible region is not unbounded, we say the LP's feasible region is bounded. Suppose an LP has a bounded feasible region. Explain why you can find the optimal solution to the LP (without an isoprofit or isocost line) by simply checking the z-values at each of the feasible region's extreme points. Why might this method fail if the LP's feasible region is unbounded?

8 Graphically find all optimal solutions to the following LP:

$$\begin{aligned}\min z &= x_1 - x_2\\ \text{s.t.}\quad x_1 + x_2 &\le 6\\ x_1 - x_2 &\ge 0\\ x_2 - x_1 &\ge 3\\ x_1, x_2 &\ge 0\end{aligned}$$

9 Graphically determine two optimal solutions to the following LP:

$$\begin{aligned}\min z &= 3x_1 + 5x_2\\ \text{s.t.}\quad 3x_1 + 2x_2 &\ge 36\\ 3x_1 + 5x_2 &\ge 45\\ x_1, x_2 &\ge 0\end{aligned}$$

Group B

10 Money manager Boris Milkem deals with French currency (the franc) and American currency (the dollar). At 12 midnight, he can buy francs by paying .25 dollars per franc and dollars by paying 3 francs per dollar. Let x_1 = number of dollars bought (by paying francs) and x_2 = number of francs bought (by paying dollars). Assume that both types of transactions take place simultaneously, and the only constraint is that at 12:01 A.M. Boris must have a nonnegative number of francs and dollars.

a Formulate an LP that enables Boris to maximize the number of dollars he has after all transactions are completed.

b Graphically solve the LP and comment on the answer.

3.4 A Diet Problem

Many LP formulations (such as Example 2 and the following diet problem) arise from situations in which a decision maker wants to minimize the cost of meeting a set of requirements.

EXAMPLE 6 Diet Problem

My diet requires that all the food I eat come from one of the four "basic food groups" (chocolate cake, ice cream, soda, and cheesecake). At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs 50¢, each scoop of chocolate ice cream costs 20¢, each bottle of cola costs 30¢, and each piece of pineapple cheesecake costs 80¢. Each day, I must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in Table 2. Formulate a linear programming model that can be used to satisfy my daily nutritional requirements at minimum cost.

Solution As always, we begin by determining the decisions that must be made by the decision maker: how much of each type of food should be eaten daily. Thus, we define the decision variables:

$$x_1 = \text{number of brownies eaten daily}$$
$$x_2 = \text{number of scoops of chocolate ice cream eaten daily}$$
$$x_3 = \text{bottles of cola drunk daily}$$
$$x_4 = \text{pieces of pineapple cheesecake eaten daily}$$

My objective is to minimize the cost of my diet. The total cost of any diet may be determined from the following relation: (total cost of diet) = (cost of brownies) + (cost of ice cream) + (cost of cola) + (cost of cheesecake). To evaluate the total cost of a diet, note that, for example,

$$\text{Cost of cola} = \left(\frac{\text{cost}}{\text{bottle of cola}}\right)\left(\begin{matrix}\text{bottles of}\\ \text{cola drunk}\end{matrix}\right) = 30x_3$$

Applying this to the other three foods, we have (in cents)

$$\text{Total cost of diet} = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

Thus, the objective function is

$$\min z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

The decision variables must satisfy the following four constraints:

Constraint 1 Daily calorie intake must be at least 500 calories.

Constraint 2 Daily chocolate intake must be at least 6 oz.

Constraint 3 Daily sugar intake must be at least 10 oz.

Constraint 4 Daily fat intake must be at least 8 oz.

TABLE 2
Nutritional Values for Diet

Type of Food	Calories	Chocolate (Ounces)	Sugar (Ounces)	Fat (Ounces)
Brownie	400	3	2	2
Chocolate ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Pineapple cheesecake (1 piece)	500	0	4	5

To express Constraint 1 in terms of the decision variables, note that (daily calorie intake) = (calories in brownies) + (calories in chocolate ice cream) + (calories in cola) + (calories in pineapple cheesecake).

The calories in the brownies consumed can be determined from

$$\text{Calories in brownies} = \left(\frac{\text{calories}}{\text{brownie}}\right)\left(\begin{matrix}\text{brownies}\\ \text{eaten}\end{matrix}\right) = 400x_1$$

Applying similar reasoning to the other three foods shows that

$$\text{Daily calorie intake} = 400x_1 + 200x_2 + 150x_3 + 500x_4$$

Constraint 1 may be expressed by

$$400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \qquad \text{(Calorie constraint)} \tag{21}$$

Constraint 2 may be expressed by

$$3x_1 + 2x_2 \geq 6 \qquad \text{(Chocolate constraint)} \tag{22}$$

Constraint 3 may be expressed by

$$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \qquad \text{(Sugar constraint)} \tag{23}$$

Constraint 4 may be expressed by

$$2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 \qquad \text{(Fat constraint)} \tag{24}$$

Finally, the sign restrictions $x_i \geq 0$ ($i = 1, 2, 3, 4$) must hold.

Combining the objective function, constraints (21)–(24), and the sign restrictions yields the following:

$$\begin{array}{llr}
\min z = 50x_1 + 20x_2 + 30x_3 + 80x_4 & & \\
\text{s.t.} \quad 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 & \text{(Calorie constraint)} & (21) \\
\quad 3x_1 + 2x_2 \geq 6 & \text{(Chocolate constraint)} & (22) \\
\quad 2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 & \text{(Sugar constraint)} & (23) \\
\quad 2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 & \text{(Fat constraint)} & (24) \\
\quad x_i \geq 0 \ (i = 1, 2, 3, 4) & \text{(Sign restrictions)} &
\end{array}$$

The optimal solution to this LP is $x_1 = x_4 = 0$, $x_2 = 3$, $x_3 = 1$, $z = 90$. Thus, the minimum-cost diet incurs a daily cost of 90¢ by eating three scoops of chocolate ice cream and drinking one bottle of cola. The optimal z-value may be obtained by substituting the optimal value of the decision variables into the objective function. This yields a total cost of $z = 3(20) + 1(30) = 90$¢. The optimal diet provides

$$\begin{aligned}
200(3) + 150(1) &= 750 \text{ calories} \\
2(3) &= 6 \text{ oz of chocolate} \\
2(3) + 4(1) &= 10 \text{ oz of sugar} \\
4(3) + 1(1) &= 13 \text{ oz of fat}
\end{aligned}$$

Thus, the chocolate and sugar constraints are binding, but the calories and fat constraints are nonbinding.

A version of the diet problem with a more realistic list of foods and nutritional requirements was one of the first LPs to be solved by computer. Stigler (1945) proposed a diet problem in which 77 types of food were available and 10 nutritional requirements (vitamin A, vitamin C, and so on) had to be satisfied. When solved by computer, the optimal solution yielded a diet consisting of corn meal, wheat flour, evaporated milk, peanut butter, lard, beef, liver, potatoes, spinach, and cabbage. Although such a diet is clearly high in vital nutrients, few people would be satisfied with it because it does not seem to meet a minimum standard of tastiness (and Stigler required that the same diet be eaten each day). The optimal solution to any LP model will reflect only those aspects of reality that are captured by the objective function and constraints. Stigler's (and our) formulation of the diet problem did not reflect people's desire for a tasty and varied diet. Integer programming has been used to plan institutional menus for a weekly or monthly period.† Menu-planning models do contain constraints that reflect tastiness and variety requirements.

PROBLEMS

Group A

1 There are three factories on the Momiss River (1, 2, and 3). Each emits two types of pollutants (1 and 2) into the river. If the waste from each factory is processed, the pollution in the river can be reduced. It costs $15 to process a ton of factory 1 waste, and each ton processed reduces the amount of pollutant 1 by 0.10 ton and the amount of pollutant 2 by 0.45 ton. It costs $10 to process a ton of factory 2 waste, and each ton processed will reduce the amount of pollutant 1 by 0.20 ton and the amount of pollutant 2 by 0.25 ton. It costs $20 to process a ton of factory 3 waste, and each ton processed will reduce the amount of pollutant 1 by 0.40 ton and the amount of pollutant 2 by 0.30 ton. The state wants to reduce the amount of pollutant 1 in the river by at least 30 tons and the amount of pollutant 2 in the river by at least 40 tons. Formulate an LP that will minimize the cost of reducing pollution by the desired amounts. Do you think that the LP assumptions (Proportionality, Additivity, Divisibility, and Certainty) are reasonable for this problem?

2‡ U.S. Labs manufactures mechanical heart valves from the heart valves of pigs. Different heart operations require valves of different sizes. U.S. Labs purchases pig valves from three different suppliers. The cost and size mix of the valves purchased from each supplier are given in Table 3. Each month, U.S. Labs places one order with each supplier. At least 500 large, 300 medium, and 300 small valves must be purchased each month. Because of limited availability of pig valves, at most 700 valves per month can be purchased from each supplier. Formulate an LP that can be used to minimize the cost of acquiring the needed valves.

TABLE 3

Supplier	Cost Per Value ($)	Percent Large	Percent Medium	Percent Small
1	5	40	40	20
2	4	30	35	35
3	3	20	20	60

3 Peg and Al Fundy have a limited food budget, so Peg is trying to feed the family as cheaply as possible. However, she still wants to make sure her family members meet their daily nutritional requirements. Peg can buy two foods. Food 1 sells for $7 per pound, and each pound contains 3 units of vitamin A and 1 unit of vitamin C. Food 2 sells for $1 per pound, and each pound contains 1 unit of each vitamin. Each day, the family needs at least 12 units of vitamin A and 6 units of vitamin C.

a Verify that Peg should purchase 12 units of food 2 each day and thus oversatisfy the vitamin C requirement by 6 units.

b Al has put his foot down and demanded that Peg fulfill the family's daily nutritional requirement exactly by obtaining precisely 12 units of vitamin A and 6 units of vitamin C. The optimal solution to the new problem will involve ingesting less vitamin C, but it will be more expensive. Why?

4 Goldilocks needs to find at least 12 lb of gold and at least 18 lb of silver to pay the monthly rent. There are two mines in which Goldilocks can find gold and silver. Each day that Goldilocks spends in mine 1, she finds 2 lb of gold and 2 lb of silver. Each day that Goldilocks spends in mine 2, she finds 1 lb of gold and 3 lb of silver. Formulate an LP to help Goldilocks meet her requirements while spending as little time as possible in the mines. Graphically solve the LP.

†Balintfy (1976).
‡Based on Hilal and Erickson (1981).

3.5 A Work-Scheduling Problem

Many applications of linear programming involve determining the minimum-cost method for satisfying workforce requirements. The following example illustrates the basic features common to many of these applications.

EXAMPLE 7 Post Office Problem

A post office requires different numbers of full-time employees on different days of the week. The number of full-time employees required on each day is given in Table 4. Union rules state that each full-time employee must work five consecutive days and then receive two days off. For example, an employee who works Monday to Friday must be off on Saturday and Sunday. The post office wants to meet its daily requirements using only full-time employees. Formulate an LP that the post office can use to minimize the number of full-time employees who must be hired.

Solution Before giving the correct formulation of this problem, let's begin by discussing an *incorrect* solution. Many students begin by defining x_i to be the number of employees working on day i (day 1 = Monday, day 2 = Tuesday, and so on). Then they reason that (number of full-time employees) = (number of employees working on Monday) + (number of employees working on Tuesday) + ⋯ + (number of employees working on Sunday). This reasoning leads to the following objective function:

$$\min z = x_1 + x_2 + \cdots + x_6 + x_7$$

To ensure that the post office has enough full-time employees working on each day, they add the constraints $x_i \geq$ (number of employees required on day i). For example, for Monday add the constraint $x_1 \geq 17$. Adding the sign restrictions $x_i \geq 0$ $(i = 1, 2, \ldots, 7)$ yields the following LP:

$$\begin{aligned}
\min z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 & \\
\text{s.t.} \quad x_1 & \geq 17 \\
x_2 & \geq 13 \\
x_3 & \geq 15 \\
x_4 & \geq 19 \\
x_5 & \geq 14 \\
x_6 & \geq 16 \\
x_7 & \geq 11 \\
x_i & \geq 0 \quad (i = 1, 2, \ldots, 7)
\end{aligned}$$

There are at least two flaws in this formulation. First, the objective function is *not* the number of full-time post office employees. The current objective function counts each employee five times, not once. For example, each employee who starts work on Monday works Monday to Friday and is included in x_1, x_2, x_3, x_4, and x_5. Second, the variables x_1, $x_2, \ldots, x_7$ are interrelated, and the interrelation between the variables is not captured by the current set of constraints. For example, some of the people who are working on Monday (the x_1 people) will be working on Tuesday. This means that x_1 and x_2 are interrelated, but our constraints do not indicate that the value of x_1 has any effect on the value of x_2.

The key to correctly formulating this problem is to realize that the post office's primary decision is not how many people are working each day but rather how many people *begin* work on each day of the week. With this in mind, we define

TABLE 4
Requirements for Post Office

Day	Number of Full-time Employees Required
1 = Monday	17
2 = Tuesday	13
3 = Wednesday	15
4 = Thursday	19
5 = Friday	14
6 = Saturday	16
7 = Sunday	11

$$x_i = \text{number of employees beginning work on day } i$$

For example, x_1 is the number of people beginning work on Monday (these people work Monday to Friday). With the variables properly defined, it is easy to determine the correct objective function and constraints. To determine the objective function, note that (number of full-time employees) = (number of employees who start work on Monday) + (number of employees who start work on Tuesday) $+\cdots+$ (number of employees who start work on Sunday). Because each employee begins work on exactly one day of the week, this expression does not double-count employees. Thus, when we correctly define the variables, the objective function is

$$\min z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

The post office must ensure that enough employees are working on each day of the week. For example, at least 17 employees must be working on Monday. Who is working on Monday? Everybody except the employees who begin work on Tuesday or on Wednesday (they get, respectively, Sunday and Monday, and Monday and Tuesday off). This means that the number of employees working on Monday is $x_1 + x_4 + x_5 + x_6 + x_7$. To ensure that at least 17 employees are working on Monday, we require that the constraint

$$x_1 + x_4 + x_5 + x_6 + x_7 \geq 17$$

be satisfied. Adding similar constraints for the other six days of the week and the sign restrictions $x_i \geq 0$ $(i = 1, 2, \ldots, 7)$ yields the following formulation of the post office's problem:

$$\begin{array}{lllr}
\min z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 & & & \\
\text{s.t.} & x_1 + x_4 + x_5 + x_6 + x_7 \geq 17 & & \text{(Monday constraint)} \\
& x_1 + x_2 + x_5 + x_6 + x_7 \geq 13 & & \text{(Tuesday constraint)} \\
& x_1 + x_2 + x_3 + x_6 + x_7 \geq 15 & & \text{(Wednesday constraint)} \\
& x_1 + x_2 + x_3 + x_4 + x_7 \geq 19 & & \text{(Thursday constraint)} \\
& x_1 + x_2 + x_3 + x_4 + x_5 \geq 14 & & \text{(Friday constraint)} \\
& x_2 + x_3 + x_4 + x_5 + x_6 \geq 16 & & \text{(Saturday constraint)} \\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq 11 & & \text{(Sunday constraint)} \\
& x_i \geq 0 \quad (i = 1, 2, \ldots, 7) & & \text{(Sign restrictions)}
\end{array}$$

The optimal solution to this LP is $z = \frac{67}{3}$, $x_1 = \frac{4}{3}$, $x_2 = \frac{10}{3}$, $x_3 = 2$, $x_4 = \frac{22}{3}$, $x_5 = 0$, $x_6 = \frac{10}{3}$, $x_7 = 5$. Because we are only allowing full-time employees, however, the variables must be integers, and the Divisibility Assumption is not satisfied. To find a reasonable answer in which all variables are integers, we could try to round the fractional variables up, yielding the feasible solution $z = 25$, $x_1 = 2$, $x_2 = 4$, $x_3 = 2$, $x_4 = 8$, $x_5 = 0$, $x_6 = 4$, $x_7 = 5$. It turns out, however, that integer programming can be used to show that an optimal solution to the post office problem is $z = 23$, $x_1 = 4$, $x_2 = 4$, $x_3 = 2$, $x_4 = 6$, $x_5 = 0$, $x_6 = 4$, $x_7 = 3$. Notice that there is no way that the optimal linear programming solution could have been rounded to obtain the optimal all-integer solution.

Baker (1974) has developed an efficient technique (that does not use linear programming) to determine the minimum number of employees required when each worker receives two consecutive days off.

If you solve this problem using LINDO, LINGO, or the Excel Solver, you may get a different workforce schedule that uses 23 employees. This shows that Example 7 has alternative optimal solutions.

Creating a Fair Schedule for Employees

The optimal solution we found requires 4 workers to start on Monday, 4 on Tuesday, 2 on Wednesday, 6 on Thursday, 4 on Saturday, and 3 on Sunday. The workers who start on Saturday will be unhappy because they never receive a weekend day off. By rotating the schedules of the employees over a 23-week period, a fairer schedule can be obtained. To see how this is done, consider the following schedule:

- weeks 1–4: start on Monday
- weeks 5–8: start on Tuesday
- weeks 9–10: start on Wednesday
- weeks 11–16: start on Thursday
- weeks 17–20: start on Saturday
- weeks 21–23: start on Sunday

Employee 1 follows this schedule for a 23-week period. Employee 2 starts with week 2 of this schedule (starting on Monday for 3 weeks, then on Tuesday for 4 weeks, and closing with 3 weeks starting on Sunday and 1 week on Monday). We continue in this fashion to generate a 23-week schedule for each employee. For example, employee 13 will have the following schedule:

- weeks 1–4: start on Thursday
- weeks 5–8: start on Saturday
- weeks 9–11: start on Sunday
- weeks 12–15: start on Monday
- weeks 16–19: start on Tuesday
- weeks 20–21: start on Wednesday
- weeks 22–23 start on Thursday

This method of scheduling treats each employee equally.

Modeling Issues

1 This example is a **static scheduling problem,** because we assume that the post office faces the same schedule each week. In reality, demands change over time, workers take vacations in the summer, and so on, so the post office does not face the same situation each week. A **dynamic scheduling problem** will be discussed in Section 3.12.

2 If you wanted to set up a weekly scheduling model for a supermarket or a fast-food restaurant, the number of variables could be very large and the computer might have difficulty finding an exact solution. In this case, **heuristic methods** can be used to find a good solution to the problem. See Love and Hoey (1990) for an example of scheduling a fast-food restaurant.

3 Our model can easily be expanded to handle part-time employees, the use of overtime, and alternative objective functions such as maximizing the number of weekend days off. (See Problems 1, 3, and 4.)

4 How did we determine the number of workers needed each day? Perhaps the post office wants to have enough employees to ensure that 95% of all letters are sorted within an hour. To determine the number of employees needed to provide adequate service, the post office would use queuing theory, which is discussed in *Stochastic Models in Operations Research: Applications and Algorithms;* and forecasting, which is discussed in Chapter 14 of this book.

Real-World Application

Krajewski, Ritzman, and McKenzie (1980) used LP to schedule clerks who processed checks at the Ohio National Bank. Their model determined the minimum-cost combination of part-time employees, full-time employees, and overtime labor needed to process each day's checks by the end of the workday (10 P.M.). The major input to their model was a forecast of the number of checks arriving at the bank each hour. This forecast was produced using multiple regression (see *Stochastic Models in Operations Research: Applications and Algorithms*). The major output of the LP was a work schedule. For example, the LP might suggest that 2 full-time employees work daily from 11 A.M. to 8 P.M., 33 part-time employees work every day from 6 P.M. to 10 P.M., and 27 part-time employees work from 6 P.M. to 10 P.M. on Monday, Tuesday, and Friday.

PROBLEMS

Group A

1 In the post office example, suppose that each full-time employee works 8 hours per day. Thus, Monday's requirement of 17 workers may be viewed as a requirement of $8(17) = 136$ hours. The post office may meet its daily labor requirements by using both full-time and part-time employees. During each week, a full-time employee works 8 hours a day for five consecutive days, and a part-time employee works 4 hours a day for five consecutive days. A full-time employee costs the post office $15 per hour, whereas a part-time employee (with reduced fringe benefits) costs the post office only $10 per hour. Union requirements limit part-time labor to 25% of weekly labor requirements. Formulate an LP to minimize the post office's weekly labor costs.

2 During each 4-hour period, the Smalltown police force requires the following number of on-duty police officers: 12 midnight to 4 A.M.—8; 4 to 8 A.M.—7; 8 A.M. to 12 noon—6; 12 noon to 4 P.M.—6; 4 to 8 P.M.—5; 8 P.M. to 12 midnight—4. Each police officer works two consecutive 4-hour shifts. Formulate an LP that can be used to minimize the number of police officers needed to meet Smalltown's daily requirements.

Group B

3 Suppose that the post office can force employees to work one day of overtime each week. For example, an employee whose regular shift is Monday to Friday can also be required to work on Saturday. Each employee is paid $50 a day for each of the first five days worked during a week and $62 for the overtime day (if any). Formulate an LP whose solution will enable the post office to minimize the cost of meeting its weekly work requirements.

4 Suppose the post office had 25 full-time employees and was not allowed to hire or fire any employees. Formulate an LP that could be used to schedule the employees in order to maximize the number of weekend days off received by the employees.

5 Each day, workers at the Gotham City Police Department work two 6-hour shifts chosen from 12 A.M. to 6 A.M., 6 A.M. to 12 P.M., 12 P.M. to 6 P.M., and 6 P.M. to 12 A.M. The following number of workers are needed during each shift: 12 A.M. to 6 A.M.—15 workers; 6 A.M. to 12 P.M.—5 workers; 12 P.M. to 6 P.M.—12 workers; 6 P.M. to 12 A.M.—6 workers. Workers whose two shifts are consecutive are paid $12 per hour; workers whose shifts are not consecutive are paid $18 per hour. Formulate an LP that can be used to minimize the cost of meeting the daily workforce demands of the Gotham City Police Department.

6 During each 6-hour period of the day, the Bloomington Police Department needs at least the number of policemen shown in Table 5. Policemen can be hired to work either 12 consecutive hours or 18 consecutive hours. Policemen are paid $4 per hour for each of the first 12 hours a day they work and are paid $6 per hour for each of the next 6 hours they work in a day. Formulate an LP that can be used to minimize the cost of meeting Bloomington's daily police requirements.

TABLE 5

Time Period	Number of Policemen Required
12 A.M.–6 A.M.	12
6 A.M.–12 P.M.	8
12 P.M.–6 P.M.	6
6 P.M.–12 A.M.	15

7 Each hour from 10 A.M. to 7 P.M., Bank One receives checks and must process them. Its goal is to process all the checks the same day they are received. The bank has 13 check-processing machines, each of which can process up to 500 checks per hour. It takes one worker to operate each machine. Bank One hires both full-time and part-time workers. Full-time workers work 10 A.M.–6 P.M., 11 A.M.–7 P.M., or Noon–8 P.M. and are paid $160 per day. Part-time workers work either 2 P.M.–7 P.M. or 3 P.M.–8 P.M. and are paid $75 per day. The number of checks received each hour is given in Table 6. In the interest of maintaining continuity, Bank One believes it must have at least three full-time workers under contract. Develop a cost-minimizing work schedule that processes all checks by 8 P.M.

TABLE 6

Time	Checks Received
10 A.M.	5,000
11 A.M.	4,000
Noon	3,000
1 P.M.	4,000
2 P.M.	2,500
3 P.M.	3,000
4 P.M.	4,000
5 P.M.	4,500
6 P.M.	3,500
7 P.M.	3,000

3.6 A Capital Budgeting Problem

In this section (and in Sections 3.7 and 3.11), we discuss how linear programming can be used to determine optimal financial decisions. This section considers a simple capital budgeting model.[†]

We first explain briefly the concept of net present value (NPV), which can be used to compare the desirability of different investments. Time 0 is the present.

Suppose investment 1 requires a cash outlay of $10,000 at time 0 and a cash outlay of $14,000 two years from now and yields a cash flow of $24,000 one year from now. Investment 2 requires a $6,000 cash outlay at time 0 and a $1,000 cash outlay two years from now and yields a cash flow of $8,000 one year from now. Which investment would you prefer?

Investment 1 has a net cash flow of

$$-10{,}000 + 24{,}000 - 14{,}000 = \$0$$

and investment 2 has a net cash flow of

$$-6{,}000 + 8{,}000 - 1{,}000 = \$1{,}000$$

On the basis of net cash flow, investment 2 is superior to investment 1. When we compare investments on the basis of net cash flow, we are assuming that a dollar received at

[†]This section is based on Weingartner (1963).

any point in time has the same value. This is not true! Suppose that there exists an investment (such as a money market fund) for which \$1 invested at a given time will yield (with certainty) $\$(1 + r)$ one year later. We call r the *annual interest rate.* Because \$1 now can be transformed into $\$(1 + r)$ one year from now, we may write

$$\$1 \text{ now} = \$(1 + r) \text{ one year from now}$$

Applying this reasoning to the $\$(1 + r)$ obtained one year from now shows that

$$\$1 \text{ now} = \$(1 + r) \text{ one year from now} = \$(1 + r)^2 \text{ two years from now}$$

and

$$\$1 \text{ now} = \$(1 + r)^k \ k \text{ years from now}$$

Dividing both sides of this equality by $(1 + r)^k$ shows that

$$\$1 \text{ received } k \text{ years from now} = \$(1 + r)^{-k} \text{ now}$$

In other words, a dollar received k years from now is equivalent to receiving $\$(1 + r)^{-k}$ now.

We can use this idea to express all cash flows in terms of time 0 dollars (this process is called *discounting cash flows to time* 0). Using discounting, we can determine the total value (in time 0 dollars) of the cash flows for any investment. The total value (in time 0 dollars) of the cash flows for any investment is called the **net present value,** or **NPV,** of the investment. The NPV of an investment is the amount by which the investment will increase the firm's value (as expressed in time 0 dollars).

Assuming that $r = 0.20$, we can compute the NPV for investments 1 and 2.

$$\begin{aligned} \text{NPV of investment 1} &= -\ 10{,}000 + \frac{24{,}000}{1 + 0.20} - \frac{14{,}000}{(1 + 0.20)^2} \\ &= \$277.78 \end{aligned}$$

This means that if a firm invested in investment 1, then the value of the firm (in time 0 dollars) would increase by \$277.78. For investment 2,

$$\begin{aligned} \text{NPV of investment 2} &= -6{,}000 + \frac{8{,}000}{1 + 0.20} - \frac{1{,}000}{(1 + 0.20)^2} \\ &= -\$27.78 \end{aligned}$$

If a firm invested in investment 2, then the value of the firm (in time 0 dollars) would be reduced by \$27.78.

Thus, the NPV concept says that investment 1 is superior to investment 2. This conclusion is contrary to the one reached by comparing the net cash flows of the two investments. Note that the comparison between investments often depends on the value of r. For example, the reader is asked to show in Problem 1 at the end of this section that for $r = 0.02$, investment 2 has a higher NPV than investment 1. Of course, our analysis assumes that the future cash flows of an investment are known with certainty.

Computing NPV with Excel

If we receive a cash flow of c_t in t years from now ($t = 1, 2, \ldots T$) and we discount cash flows at a rate r, then the NPV of our cash flows is given by

$$\sum_{t=1}^{t=T} \frac{c_t}{(1 + r)^t}$$

The basic idea is that \$1 today equals \$$(1 + r)$ a year from now, so

$$\frac{1}{1 + r} \text{ today} = \$1 \text{ a year from now}$$

The Excel function = NPV makes this computation easy. The syntax is

$$= \text{NPV}\ (r, \text{range of cash flows})$$

The formula assumes that cash flows occur at the end of the year.

Projects with NPV > 0 add value to the company, while projects with negative NPV reduce the company's value.

NPV.xls

We illustrate the computation of NPV in the file NPV.xls.

EXAMPLE 8 Computing NPV

For a discount rate of 15%, consider a project with the cash flows shown in Figure 8.

a Compute project NPV if cash flows are at the end of the year.

b Compute project NPV if cash flows are at the beginning of the year.

c Compute project NPV if cash flows are at the middle of the year.

Solution **a** We enter in cell C7 the formula

$$= \text{NPV(C1,C4:I4)}$$

and obtain \$375.06.

b Because all cash flows are received a year earlier, we multiply each cash flow's value by $(1 + 1.15)$, so the answer is obtained in C8 with formula

$$= (1 + \text{C1}) \cdot \text{C7}$$

NPV is now larger: \$431.32.

We checked this in cell D8 with the formula

$$= \text{C4} + \text{NPV(C1,D4:I4)}$$

c Because all cash flows are received six months earlier, we multiply each cash flow's value by $\sqrt{1.15}$. NPV is now computed in C9 with the formula

$$= (1.15)\text{^}0.5 \cdot \text{C7}$$

Now NPV is \$402.21.

	A	B	C	D	E	F	G	H	I
1		dr	0.15						
2									
3		Time	1	2	3	4	5	6	7
4			-400	200	600	-900	1000	250	230
5									
6									
7	end of year	end of yr.	$375.06						
8	beginning of yr.	beg. of yr.	$431.32	$431.32					
9	middle of year	middle of yr.	$402.21						

FIGURE 8

The XNPV Function

Often cash flows occur at irregular intervals. This makes it difficult to compute the NPV of these cash flows. Fortunately, the Excel XNPV function makes computing NPVs of irregularly timed cash flows a snap. To use the XNPV function, you must first have added the Analysis Toolpak. To do this, select Tools Add-Ins and check the Analysis Toolpak and Analysis Tookpak VBA boxes. Here is an example of XNPV in action.

EXAMPLE 9 Finding NPV of Nonperiodic Cash Flows

Suppose on April 8, 2001, we paid out $900. Then we receive

- $300 on 8/15/01
- $400 on 1/15/02
- $200 on 6/25/02
- $100 on 7/03/03.

If the annual discount rate is 10%, what is the NPV of these cash flows?

Solution We enter the dates (in Excel date format) in D3:D7 and the cash flows in E3:E7 (see Figure 9). Entering the formula

= XNPV(A9,E3:E7,D3:D7)

in cell D11 computes the project's NPV in terms of April 8, 2001, dollars because that is the first date chronologically. What Excel did was as follows:

1 Compute the number of years after April 8, 2001, that each date occurred. (We did this in column F). For example, August 15, 2001, is .3534 years after April 8.

2 Then discount cash flows at a rate $\left(\frac{1}{1 + \text{rate}}\right)^{\text{years after}}$. For example, the August 15, 2001, cash flow is discounted by $\left(\frac{1}{1 + .1}\right)^{.3534} = .967$.

3 We obtained Excel dates in serial number form by changing format to General.

If you want the XNPV function to determine a project's NPV in today's dollars, insert a $0 cash flow on today's date and include this row in the XNPV calculation. Excel will then return the project's NPV as of today's date.

	A	B	C	D	E	F	G
1							
2	XNPV Function		Code	Date	Cash Flow	Time	df
3			36989.00	4/8/01	-900		1
4			37118.00	8/15/01	300	0.353425	0.966876
5			37271.00	1/15/02	400	0.772603	0.929009
6			37432.00	6/25/02	200	1.213699	0.890762
7			37805.00	7/3/03	100	2.235616	0.808094
8	Rate						
9	0.1						
10				XNPV	Direct		
11				20.62822	20.628217		
12							
13				XIRR			
14				12.97%			

FIGURE 9 Example of XNPV Function

With this background information, we are ready to explain how linear programming can be applied to problems in which limited investment funds must be allocated to investment projects. Such problems are called **capital budgeting problems.**

EXAMPLE 10 Project Selection

Star Oil Company is considering five different investment opportunities. The cash outflows and net present values (in millions of dollars) are given in Table 7. Star Oil has \$40 million available for investment now (time 0); it estimates that one year from now (time 1) \$20 million will be available for investment. Star Oil may purchase any fraction of each investment. In this case, the cash outflows and NPV are adjusted accordingly. For example, if Star Oil purchases one-fifth of investment 3, then a cash outflow of $\frac{1}{5}(5) = \$1$ million would be required at time 0, and a cash outflow of $\frac{1}{5}(5) = \$1$ million would be required at time 1. The one-fifth share of investment 3 would yield an NPV of $\frac{1}{5}(16) = \$3.2$ million. Star Oil wants to maximize the NPV that can be obtained by investing in investments 1–5. Formulate an LP that will help achieve this goal. Assume that any funds left over at time 0 cannot be used at time 1.

Solution Star Oil must determine what fraction of each investment to purchase. We define

$$x_i = \text{fraction of investment } i \text{ purchased by Star Oil} \qquad (i = 1, 2, 3, 4, 5)$$

Star's goal is to maximize the NPV earned from investments. Now, (total NPV) = (NPV earned from investment 1) + (NPV earned from investment 2) + ⋯ + (NPV earned from investment 5). Note that

$$\begin{aligned}\text{NPV from investment 1} &= (\text{NPV from investment 1})(\text{fraction of investment 1 purchased})\\ &= 13x_1\end{aligned}$$

Applying analogous reasoning to investments 2–5 shows that Star Oil wants to maximize

$$z = 13x_1 + 16x_2 + 16x_3 + 14x_4 + 39x_5 \tag{25}$$

Star Oil's constraints may be expressed as follows:

Constraint 1 Star cannot invest more than \$40 million at time 0.

Constraint 2 Star cannot invest more than \$20 million at time 1.

Constraint 3 Star cannot purchase more than 100% of investment i ($i = 1, 2, 3, 4, 5$).

To express Constraint 1 mathematically, note that (dollars invested at time 0) = (dollars invested in investment 1 at time 0) + (dollars invested in investment 2 at time 0) + ⋯ + (dollars invested in investment 5 at time 0). Also, in millions of dollars,

$$\begin{aligned}\begin{matrix}\text{Dollars invested in investment 1}\\ \text{at time 0}\end{matrix} &= \begin{pmatrix}\text{dollars required for}\\ \text{investment 1 at time 0}\end{pmatrix}\begin{pmatrix}\text{fraction of}\\ \text{investment 1 purchased}\end{pmatrix}\\ &= 11x_1\end{aligned}$$

TABLE 7
Cash Flows and Net Present Value for Investments in Capital Budgeting

	Investment (\$)				
	1	2	3	4	5
Time 0 cash outflow	11	53	5	5	29
Time 1 cash outflow	3	6	5	1	34
NPV	13	16	16	14	39

Similarly, for investments 2–5,

$$\text{Dollars invested at time 0} = 11x_1 + 53x_2 + 5x_3 + 5x_4 + 29x_5$$

Then Constraint 1 reduces to

$$11x_1 + 53x_2 + 5x_3 + 5x_4 + 29x_5 \le 40 \qquad \text{(Time 0 constraint)} \tag{26}$$

Constraint 2 reduces to

$$3x_1 + 6x_2 + 5x_3 + x_4 + 34x_5 \le 20 \qquad \text{(Time 1 constraint)} \tag{27}$$

Constraints 3–7 may be represented by

$$x_i \le 1 \qquad (i = 1, 2, 3, 4, 5) \tag{28–32}$$

Combining (26)–(32) with the sign restrictions $x_i \ge 0$ ($i = 1, 2, 3, 4, 5$) yields the following LP:

$$\begin{aligned}
\max z &= 13x_1 + 16x_2 + 16x_3 + 14x_4 + 39x_5 \\
\text{s.t.} \quad & 11x_1 + 53x_2 + 5x_3 + 5x_4 + 29x_5 \le 40 \qquad \text{(Time 0 constraint)} \\
& 3x_1 + 6x_2 + 5x_3 + x_4 + 34x_5 \le 20 \qquad \text{(Time 1 constraint)} \\
& x_1 \le 1 \\
& x_2 \le 1 \\
& x_3 \le 1 \\
& x_4 \le 1 \\
& x_5 \le 1 \\
& x_i \ge 0 \qquad (i = 1, 2, 3, 4, 5)
\end{aligned}$$

The optimal solution to this LP is $x_1 = x_3 = x_4 = 1$, $x_2 = 0.201$, $x_5 = 0.288$, $z = 57.449$. Star Oil should purchase 100% of investments 1, 3, and 4; 20.1% of investment 2; and 28.8% of investment 5. A total NPV of \$57,449,000 will be obtained from these investments.

It is often impossible to purchase only a fraction of an investment without sacrificing the investment's favorable cash flows. Suppose it costs \$12 million to drill an oil well just deep enough to locate a \$30-million gusher. If there were a sole investor in this project who invested \$6 million to undertake half of the project, then he or she would lose the entire investment and receive no positive cash flows. Because, in this example, reducing the money invested by 50% reduces the return by more than 50%, this situation would violate the Proportionality Assumption.

In many capital budgeting problems, it is unreasonable to allow the x_i to be fractions: Each x_i should be restricted to 0 (not investing at all in investment i) or 1 (purchasing all of investment i). Thus, many capital budgeting problems violate the Divisibility Assumption.

A capital budgeting model that allows each x_i to be only 0 or 1 is discussed in Section 9.2.

PROBLEMS

Group A

1 Show that if $r = 0.02$, investment 2 has a larger NPV than investment 1.

2 Two investments with varying cash flows (in thousands of dollars) are available, as shown in Table 8. At time 0, \$10,000 is available for investment, and at time 1, \$7,000 is available. Assuming that $r = 0.10$, set up an LP whose solution maximizes the NPV obtained from these investments. Graphically find the optimal solution to the LP.

TABLE 8

Investment	Cash Flow (in $ Thousands) at Time 0	1	2	3
1	−6	−5	7	9
2	−8	−3	9	7

(Assume that any fraction of an investment may be purchased.)

3 Suppose that *r*, the annual interest rate, is 0.20, and that all money in the bank earns 20% interest each year (that is, after being in the bank for one year, $1 will increase to $1.20). If we place $100 in the bank for one year, what is the NPV of this transaction?

4 A company has nine projects under consideration. The NPV added by each project and the capital required by each project during the next two years is given in Table 9. All figures are in millions. For example, Project 1 will add $14 million in NPV and require expenditures of $12 million during year 1 and $3 million during year 2. Fifty million is available for projects during year 1 and $20 million is available during year 2. Assuming we may undertake a fraction of each project, how can we maximize NPV?

Group B

5[†] Finco must determine how much investment and debt to undertake during the next year. Each dollar invested reduces the NPV of the company by 10¢, and each dollar of debt increases the NPV by 50¢ (due to deductibility of interest payments). Finco can invest at most $1 million during the coming year. Debt can be at most 40% of investment. Finco now has $800,000 in cash available. All investment must be paid for from current cash or borrowed money. Set up an LP whose solution will tell Finco how to maximize its NPV. Then graphically solve the LP.

TABLE 9

	Project 1	2	3	4	5	6	7	8	9
Year 1 Outflow	12	54	6	6	30	6	48	36	18
Year 2 Outflow	3	7	6	2	35	6	4	3	3
NPV	14	17	17	15	40	12	14	10	12

3.7 Short-Term Financial Planning[‡]

LP models can often be used to aid a firm in short- or long-term financial planning (also see Section 3.11). Here we consider a simple example that illustrates how linear programming can be used to aid a corporation's short-term financial planning.[§]

EXAMPLE 11 Short-Term Financial Planning

Semicond is a small electronics company that manufactures tape recorders and radios. The per-unit labor costs, raw material costs, and selling price of each product are given in Table 10. On December 1, 2002, Semicond has available raw material that is sufficient to manufacture 100 tape recorders and 100 radios. On the same date, the company's balance sheet is as shown in Table 11, and Semicond's asset–liability ratio (called the current ratio) is 20,000/10,000 = 2.

Semicond must determine how many tape recorders and radios should be produced during December. Demand is large enough to ensure that all goods produced will be sold. All sales are on credit, however, and payment for goods produced in December will not

†Based on Myers and Pogue (1974).
‡This section covers material that may be omitted with no loss of continuity.
§This section is based on an example in Neave and Wiginton (1981).

TABLE 10
Cost Information for Semicond

	Tape Recorder	Radio
Selling price	$100	$90
Labor cost	$ 50	$35
Raw material cost	$ 30	$40

TABLE 11
Balance Sheet for Semicond

	Assets	Liabilities
Cash	$10,000	
Accounts receivable[§]	$ 3,000	
Inventory outstanding[¶]	$ 7,000	
Bank loan		$10,000

[§]Accounts receivable is money owed to Semicond by customers who have previously purchased Semicond products.

[¶]Value of December 1, 2002, inventory = 30(100) + 40(100) = $7,000.

be received until February 1, 2003. During December, Semicond will collect $2,000 in accounts receivable, and Semicond must pay off $1,000 of the outstanding loan and a monthly rent of $1,000. On January 1, 2003, Semicond will receive a shipment of raw material worth $2,000, which will be paid for on February 1, 2003. Semicond's management has decided that the cash balance on January 1, 2003, must be at least $4,000. Also, Semicond's bank requires that the current ratio at the beginning of January be at least 2. To maximize the contribution to profit from December production, (revenues to be received) − (variable production costs), what should Semicond produce during December?

Solution Semicond must determine how many tape recorders and radios should be produced during December. Thus, we define

$$x_1 = \text{number of tape recorders produced during December}$$
$$x_2 = \text{number of radios produced during December}$$

To express Semicond's objective function, note that

$$\frac{\text{Contribution to profit}}{\text{Tape recorder}} = 100 - 50 - 30 = \$20$$
$$\frac{\text{Contribution to profit}}{\text{Radio}} = 90 - 35 - 40 = \$15$$

As in the Giapetto example, this leads to the objective function

$$\max z = 20x_1 + 15x_2 \tag{33}$$

Semicond faces the following constraints:

Constraint 1 Because of limited availability of raw material, at most 100 tape recorders can be produced during December.

Constraint 2 Because of limited availability of raw material, at most 100 radios can be produced during December.

Constraint 3 Cash on hand on January 1, 2002, must be at least \$4,000.

Constraint 4 (January 1 assets)/(January 1 liabilities) $\geq$ 2 must hold.

Constraint 1 is described by

$$x_1 \leq 100 \tag{34}$$

Constraint 2 is described by

$$x_2 \leq 100 \tag{35}$$

To express Constraint 3, note that

$$\begin{aligned}
\text{January 1 cash on hand} &= \text{December 1 cash on hand} \\
&\quad + \text{accounts receivable collected during December} \\
&\quad - \text{portion of loan repaid during December} \\
&\quad - \text{December rent} - \text{December labor costs} \\
&= 10{,}000 + 2{,}000 - 1{,}000 - 1{,}000 - 50x_1 - 35x_2 \\
&= 10{,}000 - 50x_1 - 35x_2
\end{aligned}$$

Now Constraint 3 may be written as

$$10{,}000 - 50x_1 - 35x_2 \geq 4{,}000 \tag{36$'$}$$

Most computer codes require each LP constraint to be expressed in a form in which all variables are on the left-hand side and the constant is on the right-hand side. Thus, for computer solution, we should write (36′) as

$$50x_1 + 35x_2 \leq 6{,}000 \tag{36}$$

To express Constraint 4, we need to determine Semicond's January 1 cash position, accounts receivable, inventory position, and liabilities in terms of x_1 and x_2. We have already shown that

$$\text{January 1 cash position} = 10{,}000 - 50x_1 - 35x_2$$

Then

$$\begin{aligned}
\text{January 1 accounts receivable} &= \text{December 1 accounts receivable} \\
&\quad + \text{accounts receivable from December sales} \\
&\quad - \text{accounts receivable collected during December} \\
&= 3{,}000 + 100x_1 + 90x_2 - 2000 \\
&= 1{,}000 + 100x_1 + 90x_2
\end{aligned}$$

It now follows that

$$\begin{aligned}
\text{Value of January 1 inventory} &= \text{value of December 1 inventory} \\
&\quad - \text{value of inventory used in December} \\
&\quad + \text{value of inventory received on January 1} \\
&= 7{,}000 - (30x_1 + 40x_2) + 2{,}000 \\
&= 9{,}000 - 30x_1 - 40x_2
\end{aligned}$$

We can now compute the January 1 asset position:

$$\begin{aligned}
\text{January 1 asset position} &= \text{January 1 cash position} + \text{January 1 accounts receivable} \\
&\quad + \text{January 1 inventory position} \\
&= (10{,}000 - 50x_1 - 35x_2) + (1{,}000 + 100x_1 + 90x_2) \\
&\quad + (9{,}000 - 30x_1 - 40x_2) \\
&= 20{,}000 + 20x_1 + 15x_2
\end{aligned}$$

Finally,

$$\begin{aligned} \text{January 1 liabilities} &= \text{December 1 liabilities} - \text{December loan payment} \\ &\quad + \text{amount due on January 1 inventory shipment} \\ &= 10{,}000 - 1{,}000 + 2{,}000 \\ &= \$11{,}000 \end{aligned}$$

Constraint 4 may now be written as

$$\frac{20{,}000 + 20x_1 + 15x_2}{11{,}000} \geq 2$$

Multiplying both sides of this inequality by 11,000 yields

$$20{,}000 + 20x_1 + 15x_2 \geq 22{,}000$$

Putting this in a form appropriate for computer input, we obtain

$$20x_1 + 15x_2 \geq 2{,}000 \tag{37}$$

Combining (33)–(37) with the sign restrictions $x_1 \geq 0$ and $x_2 \geq 0$ yields the following LP:

$$\begin{array}{llll} \max z = 20x_1 + 15x_2 & & & \\ \text{s.t.} & x_1 & \leq 100 & \text{(Tape recorder constraint)} \\ & x_2 & \leq 100 & \text{(Radio constraint)} \\ & 50x_1 + 35x_2 & \leq 6{,}000 & \text{(Cash position constraint)} \\ & 20x_1 + 15x_2 & \geq 2{,}000 & \text{(Current ratio constraint)} \\ & x_1, x_2 & \geq 0 & \text{(Sign restrictions)} \end{array}$$

When solved graphically (or by computer), the following optimal solution is obtained: $z = 2{,}500$, $x_1 = 50$, $x_2 = 100$. Thus, Semicond can maximize the contribution of December's production to profits by manufacturing 50 tape recorders and 100 radios. This will contribute $20(50) + 15(100) = \$2{,}500$ to profits.

PROBLEMS

Group A

1 Graphically solve the Semicond problem.

2 Suppose that the January 1 inventory shipment had been valued at $7,000. Show that Semicond's LP is now infeasible.

3.8 Blending Problems

Situations in which various inputs must be blended in some desired proportion to produce goods for sale are often amenable to linear programming analysis. Such problems are called **blending problems.** The following list gives some situations in which linear programming has been used to solve blending problems.

1 Blending various types of crude oils to produce different types of gasoline and other outputs (such as heating oil)

2 Blending various chemicals to produce other chemicals

3 Blending various types of metal alloys to produce various types of steels

4 Blending various livestock feeds in an attempt to produce a minimum-cost feed mixture for cattle

5 Mixing various ores to obtain ore of a specified quality

6 Mixing various ingredients (meat, filler, water, and so on) to produce a product like bologna

7 Mixing various types of papers to produce recycled paper of varying quality

The following example illustrates the key ideas that are used in formulating LP models of blending problems.

EXAMPLE 12 Oil Blending

Sunco Oil manufactures three types of gasoline (gas 1, gas 2, and gas 3). Each type is produced by blending three types of crude oil (crude 1, crude 2, and crude 3). The sales price per barrel of gasoline and the purchase price per barrel of crude oil are given in Table 12. Sunco can purchase up to 5,000 barrels of each type of crude oil daily.

The three types of gasoline differ in their octane rating and sulfur content. The crude oil blended to form gas 1 must have an average octane rating of at least 10 and contain at most 1% sulfur. The crude oil blended to form gas 2 must have an average octane rating of at least 8 and contain at most 2% sulfur. The crude oil blended to form gas 3 must have an octane rating of at least 6 and contain at most 1% sulfur. The octane rating and the sulfur content of the three types of oil are given in Table 13. It costs $4 to transform one barrel of oil into one barrel of gasoline, and Sunco's refinery can produce up to 14,000 barrels of gasoline daily.

Sunco's customers require the following amounts of each gasoline: gas 1—3,000 barrels per day; gas 2—2,000 barrels per day; gas 3—1,000 barrels per day. The company considers it an obligation to meet these demands. Sunco also has the option of advertising to stimulate demand for its products. Each dollar spent daily in advertising a particular type of gas increases the daily demand for that type of gas by 10 barrels. For example, if Sunco decides to spend $20 daily in advertising gas 2, then the daily demand for gas 2 will increase by $20(10) = 200$ barrels. Formulate an LP that will enable Sunco to maximize daily profits (profits = revenues − costs).

Solution Sunco must make two types of decisions: first, how much money should be spent in advertising each type of gas, and second, how to blend each type of gasoline from the three types of crude oil available. For example, Sunco must decide how many barrels of crude 1 should be used to produce gas 1. We define the decision variables

$$a_i = \text{dollars spent daily on advertising gas } i \qquad (i = 1, 2, 3)$$

$$x_{ij} = \text{barrels of crude oil } i \text{ used daily to produce gas } j \qquad (i = 1, 2, 3;\ j = 1, 2, 3)$$

For example, x_{21} is the number of barrels of crude 2 used each day to produce gas 1.

TABLE 12
Gas and Crude Oil Prices for Blending

Gas	Sales Price per Barrel ($)	Crude	Purchase Price per Barrel ($)
1	70	1	45
2	60	2	35
3	50	3	25

TABLE 13
Octane Ratings and Sulfur Requirements for Blending

Crude	Octane Rating	Sulfur Content (%)
1	12	0.5
2	6	2.0
3	8	3.0

Knowledge of these variables is sufficient to determine Sunco's objective function and constraints, but before we do this, we note that the definition of the decision variables implies that

$$\begin{aligned} x_{11} + x_{12} + x_{13} &= \text{barrels of crude 1 used daily} \\ x_{21} + x_{22} + x_{23} &= \text{barrels of crude 2 used daily} \\ x_{31} + x_{32} + x_{33} &= \text{barrels of crude 3 used daily} \end{aligned} \tag{38}$$

$$\begin{aligned} x_{11} + x_{21} + x_{31} &= \text{barrels of gas 1 produced daily} \\ x_{12} + x_{22} + x_{32} &= \text{barrels of gas 2 produced daily} \\ x_{13} + x_{23} + x_{33} &= \text{barrels of gas 3 produced daily} \end{aligned} \tag{39}$$

To simplify matters, let's assume that gasoline cannot be stored, so it must be sold on the day it is produced. This implies that for $i = 1, 2, 3$, the amount of gas i produced daily should equal the daily demand for gas i. Suppose that the amount of gas i produced daily exceeded the daily demand. Then we would have incurred unnecessary purchasing and production costs. On the other hand, if the amount of gas i produced daily is less than the daily demand for gas i, then we are failing to meet mandatory demands or incurring unnecessary advertising costs.

We are now ready to determine Sunco's objective function and constraints. We begin with Sunco's objective function. From (39),

$$\begin{aligned} \text{Daily revenues from gas sales} = {} & 70(x_{11} + x_{21} + x_{31}) + 60(x_{12} + x_{22} + x_{32}) \\ & + 50(x_{13} + x_{23} + x_{33}) \end{aligned}$$

From (38),

$$\begin{aligned} \text{Daily cost of purchasing crude oil} = {} & 45(x_{11} + x_{12} + x_{13}) + 35(x_{21} + x_{22} + x_{23}) \\ & + 25(x_{31} + x_{32} + x_{33}) \end{aligned}$$

Also,

$$\begin{aligned} \text{Daily advertising costs} &= a_1 + a_2 + a_3 \\ \text{Daily production costs} &= 4(x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33}) \end{aligned}$$

Then,

$$\begin{aligned} \text{Daily profit} = {} & \text{daily revenue from gas sales} \\ & - \text{daily cost of purchasing crude oil} \\ & - \text{daily advertising costs} - \text{daily production costs} \\ = {} & (70 - 45 - 4)x_{11} + (60 - 45 - 4)x_{12} + (50 - 45 - 4)x_{13} \\ & + (70 - 35 - 4)x_{21} + (60 - 35 - 4)x_{22} + (50 - 35 - 4)x_{23} \\ & + (70 - 25 - 4)x_{31} + (60 - 25 - 4)x_{32} \\ & + (50 - 25 - 4)x_{33} - a_1 - a_2 - a_3 \end{aligned}$$

Thus, Sunco's goal is to maximize

$$z = 21x_{11} + 11x_{12} + x_{13} + 31x_{21} + 21x_{22} + 11x_{23} + 41x_{31} + 31x_{32} + 21x_{33} - a_1 - a_2 - a_3 \tag{40}$$

Regarding Sunco's constraints, we see that the following 13 constraints must be satisfied:

Constraint 1 Gas 1 produced daily should equal its daily demand.

Constraint 2 Gas 2 produced daily should equal its daily demand.

Constraint 3 Gas 3 produced daily should equal its daily demand.

Constraint 4 At most 5,000 barrels of crude 1 can be purchased daily.

Constraint 5 At most 5,000 barrels of crude 2 can be purchased daily.

Constraint 6 At most 5,000 barrels of crude 3 can be purchased daily.

Constraint 7 Because of limited refinery capacity, at most 14,000 barrels of gasoline can be produced daily.

Constraint 8 Crude oil blended to make gas 1 must have an average octane level of at least 10.

Constraint 9 Crude oil blended to make gas 2 must have an average octane level of at least 8.

Constraint 10 Crude oil blended to make gas 3 must have an average octane level of at least 6.

Constraint 11 Crude oil blended to make gas 1 must contain at most 1% sulfur.

Constraint 12 Crude oil blended to make gas 2 must contain at most 2% sulfur.

Constraint 13 Crude oil blended to make gas 3 must contain at most 1% sulfur.

To express Constraint 1 in terms of decision variables, note that

$$\text{Daily demand for gas 1} = 3{,}000 + \text{gas 1 demand generated by advertising}$$

$$\text{Gas 1 demand generated by advertising} = \left(\frac{\text{gas 1 demand}}{\text{dollar spent}}\right)\left(\begin{matrix}\text{dollars}\\ \text{spent}\end{matrix}\right) = 10a_1{}^{\dagger}$$

Thus, daily demand for gas 1 $= 3{,}000 + 10a_1$. Constraint 1 may now be written as

$$x_{11} + x_{21} + x_{31} = 3{,}000 + 10a_1 \tag{41$'$}$$

which we rewrite as

$$x_{11} + x_{21} + x_{31} - 10a_1 = 3{,}000 \tag{41}$$

Constraint 2 is expressed by

$$x_{12} + x_{22} + x_{32} - 10a_2 = 2{,}000 \tag{42}$$

†Many students believe that gas 1 demand generated by advertising should be written as $\frac{1}{10}a_1$. Analyzing the units of this term will show that this is not correct. $\frac{1}{10}$ has units of dollars spent per barrel of demand, and a_1 has units of dollars spent. Thus, the term $\frac{1}{10}a_1$ would have units of (dollars spent)^2 per barrel of demand. This cannot be correct!

Constraint 3 is expressed by

$$x_{13} + x_{23} + x_{33} - 10a_3 = 1{,}000 \tag{43}$$

From (38), Constraint 4 reduces to

$$x_{11} + x_{12} + x_{13} \le 5{,}000 \tag{44}$$

Constraint 5 reduces to

$$x_{21} + x_{22} + x_{23} \le 5{,}000 \tag{45}$$

Constraint 6 reduces to

$$x_{31} + x_{32} + x_{33} \le 5{,}000 \tag{46}$$

Note that

$$\begin{aligned}\text{Total gas produced} &= \text{gas 1 produced} + \text{gas 2 produced} + \text{gas 3 produced}\\ &= (x_{11} + x_{21} + x_{31}) + (x_{12} + x_{22} + x_{32}) + (x_{13} + x_{23} + x_{33})\end{aligned}$$

Then Constraint 7 becomes

$$x_{11} + x_{21} + x_{31} + x_{12} + x_{22} + x_{32} + x_{13} + x_{23} + x_{33} \le 14{,}000 \tag{47}$$

To express Constraints 8–10, we must be able to determine the "average" octane level in a mixture of different types of crude oil. We assume that the octane levels of different crudes blend linearly. For example, if we blend two barrels of crude 1, three barrels of crude 2, and one barrel of crude 3, the average octane level in this mixture would be

$$\frac{\text{Total octane value in mixture}}{\text{Number of barrels in mixture}} = \frac{12(2) + 6(3) + 8(1)}{2 + 3 + 1} = \frac{50}{6} = 8\frac{1}{3}$$

Generalizing, we can express Constraint 8 by

$$\frac{\text{Total octane value in gas 1}}{\text{Gas 1 in mixture}} = \frac{12x_{11} + 6x_{21} + 8x_{31}}{x_{11} + x_{21} + x_{31}} \ge 10 \tag{48$'$}$$

Unfortunately, (48′) is not a linear inequality. To transform (48′) into a linear inequality, all we have to do is multiply both sides by the denominator of the left-hand side. The resulting inequality is

$$12x_{11} + 6x_{21} + 8x_{31} \ge 10(x_{11} + x_{21} + x_{31})$$

which may be rewritten as

$$2x_{11} - 4x_{21} - 2x_{31} \ge 0 \tag{48}$$

Similarly, Constraint 9 yields

$$\frac{12x_{12} + 6x_{22} + 8x_{32}}{x_{12} + x_{22} + x_{32}} \ge 8$$

Multiplying both sides of this inequality by $x_{12} + x_{22} + x_{32}$ and simplifying yields

$$4x_{12} - 2x_{22} \ge 0 \tag{49}$$

Because each type of crude oil has an octane level of 6 or higher, whatever we blend to manufacture gas 3 will have an average octane level of at least 6. This means that any values of the variables will satisfy Constraint 10. To verify this, we may express Constraint 10 by

$$\frac{12x_{13} + 6x_{23} + 8x_{33}}{x_{13} + x_{23} + x_{33}} \ge 6$$

Multiplying both sides of this inequality by $x_{13} + x_{23} + x_{33}$ and simplifying, we obtain

$$6x_{13} + 2x_{33} \geq 0 \tag{50}$$

Because $x_{13} \geq 0$ and $x_{33} \geq 0$ are always satisfied, (50) will automatically be satisfied and thus need not be included in the model. A constraint such as (50) that is implied by other constraints in the model is said to be a **redundant constraint** and need not be included in the formulation.

Constraint 11 may be written as

$$\frac{\text{Total sulfur in gas 1 mixture}}{\text{Number of barrels in gas 1 mixture}} \leq 0.01$$

Then, using the percentages of sulfur in each type of oil, we see that

$$\begin{aligned}\text{Total sulfur in gas 1 mixture} &= \text{Sulfur in oil 1 used for gas 1}\\ &\quad + \text{sulfur in oil 2 used for gas 1}\\ &\quad + \text{sulfur in oil 3 used for gas 1}\\ &= 0.005x_{11} + 0.02x_{21} + 0.03x_{31}\end{aligned}$$

Constraint 11 may now be written as

$$\frac{0.005x_{11} + 0.02x_{21} + 0.03x_{31}}{x_{11} + x_{21} + x_{31}} \leq 0.01$$

Again, this is not a linear inequality, but we can multiply both sides of the inequality by $x_{11} + x_{21} + x_{31}$ and simplify, obtaining

$$-0.005x_{11} + 0.01x_{21} + 0.02x_{31} \leq 0 \tag{51}$$

Similarly, Constraint 12 is equivalent to

$$\frac{0.005x_{12} + 0.02x_{22} + 0.03x_{32}}{x_{12} + x_{22} + x_{32}} \leq 0.02$$

Multiplying both sides of this inequality by $x_{12} + x_{22} + x_{32}$ and simplifying yields

$$-0.015x_{12} + 0.01x_{32} \leq 0 \tag{52}$$

Finally, Constraint 13 is equivalent to

$$\frac{0.005x_{13} + 0.02x_{23} + 0.03x_{33}}{x_{13} + x_{23} + x_{33}} \leq 0.01$$

Multiplying both sides of this inequality by $x_{13} + x_{23} + x_{33}$ and simplifying yields the LP constraint

$$-0.005x_{13} + 0.01x_{23} + 0.02x_{33} \leq 0 \tag{53}$$

Combining (40)–(53), except the redundant constraint (50), with the sign restrictions $x_{ij} \geq 0$ and $a_i \geq 0$ yields an LP that may be expressed in tabular form (see Table 14). In Table 14, the first row (max) represents the objective function, the second row represents the first constraint, and so on. When solved on a computer, an optimal solution to Sunco's LP is found to be

$$z = 287{,}500$$

$$\begin{array}{lll} x_{11} = 2222.22 & x_{12} = 2111.11 & x_{13} = 666.67\\ x_{21} = 444.44 & x_{22} = 4222.22 & x_{23} = 333.34\\ x_{31} = 333.33 & x_{32} = 3166.67 & x_{33} = 0\end{array}$$

$$a_1 = 0 \qquad a_2 = 750 \qquad a_3 = 0$$

TABLE 14
Objective Function and Constraints for Blending

x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	a_1	a_2	a_3	
21	11	1	31	21	11	41	31	21	−1	−1	−1	(max)
1	0	0	1	0	0	1	0	0	−10	0	0	= 3,000
0	1	0	0	1	0	0	1	0	0	−10	0	= 2,000
0	0	1	0	0	1	0	0	1	0	0	−10	= 1,000
1	1	1	0	0	0	0	0	0	0	0	0	≤ 5,000
0	0	0	1	1	1	0	0	0	0	0	0	≤ 5,000
0	0	0	0	0	0	1	1	1	0	0	0	≤ 5,000
1	1	1	1	1	1	1	1	1	0	0	0	≤ 14,000
2	0	0	−4	0	0	−2	0	0	0	0	0	≥ 0
0	4	0	0	−2	0	0	0	0	0	0	0	≥ 0
−0.005	0	0	0.01	0	0	0.02	0	0	0	0	0	≤ 0
0	−0.015	0	0	0	0	0	0.01	0	0	0	0	≤ 0
0	0	−0.005	0	0	0.01	0	0	0.02	0	0	0	≤ 0

Thus, Sunco should produce $x_{11} + x_{21} + x_{31} = 3{,}000$ barrels of gas 1, using 2222.22 barrels of crude 1, 444.44 barrels of crude 2, and 333.33 barrels of crude 3. The firm should produce $x_{12} + x_{22} + x_{32} = 9{,}500$ barrels of gas 2, using 2,111.11 barrels of crude 1, 4222.22 barrels of crude 2, and 3,166.67 barrels of crude 3. Sunco should also produce $x_{13} + x_{23} + x_{33} = 1{,}000$ barrels of gas 3, using 666.67 barrels of crude 1 and 333.34 barrels of crude 2. The firm should also spend $750 on advertising gas 2. Sunco will earn a profit of $287,500.

Observe that although gas 1 appears to be most profitable, we stimulate demand for gas 2, not gas 1. The reason for this is that given the quality (with respect to octane level and sulfur content) of the available crude, it is difficult to produce gas 1. Therefore, Sunco can make more money by producing more of the lower-quality gas 2 than by producing extra quantities of gas 1.

Modeling Issues

1 We have assumed that the quality level of a mixture is a **linear** function of each input used in the mixture. For example, we have assumed that if gas 3 is made with $\frac{2}{3}$ crude 1 and $\frac{1}{3}$ crude 2, then octane level for gas 3 $= (\frac{2}{3}) \cdot$ (octane level for crude 1) $+ (\frac{1}{3}) \cdot$ (octane level for crude 2). If the octane level of a gas is not a linear function of the fraction of each input used to produce the gas, then we no longer have a linear programming problem; we have a **nonlinear programming** problem. For example, let g_{i3} = fraction of gas 3 made with oil i. Suppose that the octane level for gas 3 is given by gas 3 octane level $= g_{13}^{.5} \cdot$ (oil 1 octane level) $+ g_{23}^{.4} \cdot$ (oil 2 octane level) $+ g_{33}^{.3} \cdot$ (oil 3 octane level). Then we do not have an LP problem. The reason for this is that the octane level of gas 3 is not a linear function of g_{13}, g_{23}, and g_{33}. We discuss nonlinear programming in Chapter 11.

2 In reality, a company using a blending model would run the model periodically (each day, say) and set production on the basis of the current inventory of inputs and current demand forecasts. Then the forecast levels and input levels would be updated, and the model would be run again to determine the next day's production.

Real-World Applications

Blending at Texaco

Texaco (see Dewitt et al., 1980) uses a nonlinear programming model (OMEGA) to plan and schedule its blending applications. The company's model is nonlinear because blend volatilities and octanes are nonlinear functions of the amount of each input used to produce a particular gasoline.

Blending in the Steel Industry

Fabian (1958) describes a complex LP model that can be used to optimize the production of iron and steel. For each product produced there are several blending constraints. For example, basic pig iron must contain at most 1.5% silicon, at most .05% sulphur, between .11% and .90% phosphorus, between .4% and 2% manganese, and between 4.1% and 4.4% carbon. See Problem 6 (in the Review Problems section) for a simple example of blending in the steel industry.

Blending in the Oil Industry

Many oil companies use LP to optimize their refinery operations. Problem 14 contains an example (based on Magoulas and Marinos-Kouris [1988]) of a blending model that can be used to maximize a refinery's profit.

PROBLEMS

Group A

1 You have decided to enter the candy business. You are considering producing two types of candies: Slugger Candy and Easy Out Candy, both of which consist solely of sugar, nuts, and chocolate. At present, you have in stock 100 oz of sugar, 20 oz of nuts, and 30 oz of chocolate. The mixture used to make Easy Out Candy must contain at least 20% nuts. The mixture used to make Slugger Candy must contain at least 10% nuts and 10% chocolate. Each ounce of Easy Out Candy can be sold for 25¢ , and each ounce of Slugger Candy for 20¢. Formulate an LP that will enable you to maximize your revenue from candy sales.

2 O.J. Juice Company sells bags of oranges and cartons of orange juice. O.J. grades oranges on a scale of 1 (poor) to 10 (excellent). O.J. now has on hand 100,000 lb of grade 9 oranges and 120,000 lb of grade 6 oranges. The average quality of oranges sold in bags must be at least 7, and the average quality of the oranges used to produce orange juice must be at least 8. Each pound of oranges that is used for juice yields a revenue of $1.50 and incurs a variable cost (consisting of labor costs, variable overhead costs, inventory costs, and so on) of $1.05. Each pound of oranges sold in bags yields a revenue of 50¢ and incurs a variable cost of 20¢. Formulate an LP to help O.J. maximize profit.

3 A bank is attempting to determine where its assets should be invested during the current year. At present, $500,000 is available for investment in bonds, home loans, auto loans, and personal loans. The annual rate of return on each type of investment is known to be: bonds, 10%; home loans, 16%; auto loans, 13%; personal loans, 20%. To ensure that the bank's portfolio is not too risky, the bank's investment manager has placed the following three restrictions on the bank's portfolio:

a The amount invested in personal loans cannot exceed the amount invested in bonds.

b The amount invested in home loans cannot exceed the amount invested in auto loans.

c No more than 25% of the total amount invested may be in personal loans.

The bank's objective is to maximize the annual return on its investment portfolio. Formulate an LP that will enable the bank to meet this goal.

4 Young MBA Erica Cudahy may invest up to $1,000. She can invest her money in stocks and loans. Each dollar invested in stocks yields 10¢ profit, and each dollar invested in a loan yields 15¢ profit. At least 30% of all money invested must be in stocks, and at least $400 must be in loans. Formulate an LP that can be used to maximize total profit earned from Erica's investment. Then graphically solve the LP.

5 Chandler Oil Company has 5,000 barrels of oil 1 and 10,000 barrels of oil 2. The company sells two products: gasoline and heating oil. Both products are produced by combining oil 1 and oil 2. The quality level of each oil is as follows: oil 1—10; oil 2—5. Gasoline must have an average quality level of at least 8, and heating oil at least 6. Demand for each product must be created by advertising. Each dollar spent advertising gasoline creates 5 barrels of demand and each spent on heating oil creates 10 barrels of demand. Gasoline is sold for $25 per barrel, heating oil for $20. Formulate an LP to help Chandler maximize profit. Assume that no oil of either type can be purchased.

6 Bullco blends silicon and nitrogen to produce two types of fertilizers. Fertilizer 1 must be at least 40% nitrogen and sells for $70/lb. Fertilizer 2 must be at least 70% silicon and sells for $40/lb. Bullco can purchase up to 80 lb of nitrogen at $15/lb and up to 100 lb of silicon at $10/lb. Assuming that all fertilizer produced can be sold, formulate an LP to help Bullco maximize profits.

7 Eli Daisy uses chemicals 1 and 2 to produce two drugs. Drug 1 must be at least 70% chemical 1, and drug 2 must be at least 60% chemical 2. Up to 40 oz of drug 1 can be sold at $6 per oz; up to 30 oz of drug 2 can be sold at $5 per oz. Up to 45 oz of chemical 1 can be purchased at $6 per oz, and up to 40 oz of chemical 2 can be purchased at $4 per oz. Formulate an LP that can be used to maximize Daisy's profits.

8 Highland's TV-Radio Store must determine how many TVs and radios to keep in stock. A TV requires 10 sq ft of floorspace, whereas a radio requires 4 sq ft; 200 sq ft of floorspace is available. A TV will earn Highland $60 in profits, and a radio will earn $20. The store stocks only TVs and radios. Marketing requirements dictate that at least 60% of all appliances in stock be radios. Finally, a TV ties up $200 in capital, and a radio, $50. Highland wants to have at most $3,000 worth of capital tied up at any time. Formulate an LP that can be used to maximize Highland's profit.

9 Linear programming models are used by many Wall Street firms to select a desirable bond portfolio. The following is a simplified version of such a model. Solodrex is considering investing in four bonds; $1,000,000 is available for investment. The expected annual return, the worst-case annual return on each bond, and the "duration" of each bond are given in Table 15. The duration of a bond is a measure of the bond's sensitivity to interest rates. Solodrex wants to maximize the expected return from its bond investments, subject to three constraints.

Constraint 1 The worst-case return of the bond portfolio must be at least 8%.

Constraint 2 The average duration of the portfolio must be at most 6. For example, a portfolio that invested $600,000 in bond 1 and $400,000 in bond 4 would have an average duration of

$$\frac{600{,}000(3) + 400{,}000(9)}{1{,}000{,}000} = 5.4$$

Constraint 3 Because of diversification requirements, at most 40% of the total amount invested can be invested in a single bond.

Formulate an LP that will enable Solodrex to maximize the expected return on its investment.

10 Coalco produces coal at three mines and ships it to four customers. The cost per ton of producing coal, the ash and sulfur content (per ton) of the coal, and the production capacity (in tons) for each mine are given in Table 16. The number of tons of coal demanded by each customer are given in Table 17.

The cost (in dollars) of shipping a ton of coal from a mine to each customer is given in Table 18. It is required that the total amount of coal shipped contain at most 5% ash and at most 4% sulfur. Formulate an LP that minimizes the cost of meeting customer demands.

11 Eli Daisy produces the drug Rozac from four chemicals. Today they must produce 1,000 lb of the drug. The three active ingredients in Rozac are A, B, and C. By weight, at least 8% of Rozac must consist of A, at least 4% of B, and at least 2% of C. The cost per pound of each chemical and the amount of each active ingredient in 1 lb of each chemical are given in Table 19.

It is necessary that at least 100 lb of chemical 2 be used. Formulate an LP whose solution would determine the cheapest way of producing today's batch of Rozac.

TABLE 15

Bond	Expected Return (%)	Worst-Case Return (%)	Duration
1	13	6%	3
2	8	8%	4
3	12	10%	7
4	14	9%	9

TABLE 16

Mine	Production Cost ($)	Capacity	Ash Content (Tons)	Sulfur Content (Tons)
1	50	120	.08	.05
2	55	100	.06	.04
3	62	140	.04	.03

TABLE 17

Customer 1	Customer 2	Customer 3	Customer 4
80	70	60	40

TABLE 18

	Customer			
Mine	1	2	3	4
1	4	6	8	12
2	9	6	7	11
3	8	12	3	5

TABLE 19

Chemical	Cost ($ per Lb)	A	B	C
1	8	.03	.02	.01
2	10	.06	.04	.01
3	11	.10	.03	.04
4	14	.12	.09	.04

12 (A spreadsheet might be helpful on this problem.) The *risk index* of an investment can be obtained from return on investment (ROI) by taking the percentage of change in the value of the investment (in absolute terms) for each year, and averaging them.

Suppose you are trying to determine what percentage of your money should be invested in T-bills, gold, and stocks. In Table 20 (or File Inv68.xls) you are given the annual returns (change in value) for these investments for the years 1968–1988. Let the risk index of a portfolio be the weighted (according to the fraction of your money assigned to each investment) average of the risk index of each individual investment. Suppose that the amount of each investment must be between 20% and 50% of the total invested. You would like the risk index of your portfolio to equal .15, and your goal is to maximize the expected return on your portfolio. Formulate an LP whose solution will maximize the expected return on your portfolio, subject to the given constraints. Use the average return earned by each investment during the years 1968–1988 as your estimate of expected return.†

Group B

13 The owner of Sunco does not believe that our LP optimal solution will maximize daily profit. He reasons, "We have 14,000 barrels of daily refinery capacity, but your optimal solution produces only 13,500 barrels. Therefore, it cannot be maximizing profit." How would you respond?

14 Oilco produces two products: regular and premium gasoline. Each product contains .15 gram of lead per liter. The two products are produced from six inputs: reformate, fluid catalytic cracker gasoline (FCG), isomerate (ISO), polymer (POL), MTBE (MTB), and butane (BUT). Each input has four attributes:

Attribute 1 Research octane number (RON)
Attribute 2 RVP
Attribute 3 ASTM volatility at 70°C
Attribute 4 ASTM volatility at 130°C

The attributes and daily availability (in liters) of each input are given in Table 21.

The requirements for each output are given in Table 22.

The daily demand (in thousands of liters) for each product must be met, but more can be produced if desired. The RON and ASTM requirements are minimums. Regular gasoline sells for 29.49¢/liter, premium gasoline for 31.43¢. Before being ready for sale, .15 gram/liter of lead must be removed from each product. The cost of removing .1 gram/liter is 8.5¢. At most 38% of each type of gasoline can consist of FCG. Formulate and solve an LP whose solution will tell Oilco how to maximize their daily profit.‡

TABLE 20

Year	Stocks	Gold	T-Bills
1968	11	11	5
1969	−9	8	7
1970	4	−14	7
1971	14	14	4
1972	19	44	4
1973	−15	66	7
1974	−27	64	8
1975	37	0	6
1976	24	−22	5
1977	−7	18	5
1978	7	31	7
1979	19	59	10
1980	33	99	11
1981	−5	−25	15
1982	22	4	11
1983	23	−11	9
1984	6	−15	10
1985	32	−12	8
1986	19	16	6
1987	5	22	5
1988	17	−2	6

TABLE 21

	Availability	RON	RVP	ASTM(70)	ASTM(130)
Reformate	15,572	98.9	7.66	−5	46
FCG	15,434	93.2	9.78	57	103
ISO	6,709	86.1	29.52	107	100
POL	1,190	97	14.51	7	73
MTB	748	117	13.45	98	100
BUT	Unlimited	98	166.99	130	100

TABLE 22

	Demand	RON	RVP	ASTM(70)	ASTM(130)
Regular	9.8	90	21.18	10	50
Premium	30	96	21.18	10	50

†Based on Chandy (1987).

‡Based on Magoulas and Marinos-Kouris (1988).

3.9 Production Process Models

We now explain how to formulate an LP model of a simple production process.[†] The key step is to determine how the outputs from a later stage of the process are related to the outputs from an earlier stage.

EXAMPLE 13 Brute Production Process

Rylon Corporation manufactures Brute and Chanelle perfumes. The raw material needed to manufacture each type of perfume can be purchased for \$3 per pound. Processing 1 lb of raw material requires 1 hour of laboratory time. Each pound of processed raw material yields 3 oz of Regular Brute Perfume and 4 oz of Regular Chanelle Perfume. Regular Brute can be sold for \$7/oz and Regular Chanelle for \$6/oz. Rylon also has the option of further processing Regular Brute and Regular Chanelle to produce Luxury Brute, sold at \$18/oz, and Luxury Chanelle, sold at \$14/oz. Each ounce of Regular Brute processed further requires an additional 3 hours of laboratory time and \$4 processing cost and yields 1 oz of Luxury Brute. Each ounce of Regular Chanelle processed further requires an additional 2 hours of laboratory time and \$4 processing cost and yields 1 oz of Luxury Chanelle. Each year, Rylon has 6,000 hours of laboratory time available and can purchase up to 4,000 lb of raw material. Formulate an LP that can be used to determine how Rylon can maximize profits. Assume that the cost of the laboratory hours is a fixed cost.

Solution Rylon must determine how much raw material to purchase and how much of each type of perfume should be produced. We therefore define our decision variables to be

x_1 = number of ounces of Regular Brute sold annually
x_2 = number of ounces of Luxury Brute sold annually
x_3 = number of ounces of Regular Chanelle sold annually
x_4 = number of ounces of Luxury Chanelle sold annually
x_5 = number of pounds of raw material purchased annually

Rylon wants to maximize

$$\begin{aligned}\text{Contribution to profit} &= \text{revenues from perfume sales} - \text{processing costs}\\ &\quad - \text{costs of purchasing raw material}\\ &= 7x_1 + 18x_2 + 6x_3 + 14x_4 - (4x_2 + 4x_4) - 3x_5\\ &= 7x_1 + 14x_2 + 6x_3 + 10x_4 - 3x_5\end{aligned}$$

Thus, Rylon's objective function may be written as

$$\max z = 7x_1 + 14x_2 + 6x_3 + 10x_4 - 3x_5 \tag{54}$$

Rylon faces the following constraints:

Constraint 1 No more than 4,000 lb of raw material can be purchased annually.

Constraint 2 No more than 6,000 hours of laboratory time can be used each year.

Constraint 1 is expressed by

$$x_5 \le 4{,}000 \tag{55}$$

[†]This section is based on Hartley (1971).

To express Constraint 2, note that

$$\begin{aligned}\text{Total lab time used annually} &= \text{time used annually to process raw material}\\ &\quad + \text{time used annually to process Luxury Brute}\\ &\quad + \text{time used annually to process Luxury Chanelle}\\ &= x_5 + 3x_2 + 2x_4\end{aligned}$$

Then Constraint 2 becomes

$$3x_2 + 2x_4 + x_5 \le 6{,}000 \tag{56}$$

After adding the sign restrictions $x_i \ge 0$ ($i = 1, 2, 3, 4, 5$), many students claim that Rylon should solve the following LP:

$$\begin{aligned}\max z &= 7x_1 + 14x_2 + 6x_3 + 10x_4 - 3x_5\\ \text{s.t.} \quad & x_5 \le 4{,}000\\ & 3x_2 + 2x_4 + x_5 \le 6{,}000\\ & x_i \ge 0 \qquad (i = 1, 2, 3, 4, 5)\end{aligned}$$

This formulation is incorrect. Observe that the variables x_1 and x_3 do not appear in any of the constraints. This means that any point with $x_2 = x_4 = x_5 = 0$ and x_1 and x_3 very large is in the feasible region. Points with x_1 and x_3 large can yield arbitrarily large profits. Thus, this LP is unbounded. Our mistake is that the current formulation does not indicate that the amount of raw material purchased determines the amount of Brute and Chanelle that is available for sale or further processing. More specifically, from Figure 10 (and the fact that 1 oz of processed Brute yields exactly 1 oz of Luxury Brute), it follows that

$$\begin{aligned}&\text{Ounces of Regular Brute Sold} + \text{ounces of Luxury Brute sold}\\ &\quad = \left(\frac{\text{ounces of Brute produced}}{\text{pound of raw material}}\right)\left(\text{pounds of raw material purchased}\right)\\ &\quad = 3x_5\end{aligned}$$

This relation is reflected in the constraint

$$x_1 + x_2 = 3x_5 \qquad \text{or} \qquad x_1 + x_2 - 3x_5 = 0 \tag{57}$$

Similarly, from Figure 10 it is clear that

$$\text{Ounces of Regular Chanelle sold} + \text{ounces of Luxury Chanelle sold} = 4x_5$$

This relation yields the constraint

$$x_3 + x_4 = 4x_5 \qquad \text{or} \qquad x_3 + x_4 - 4x_5 = 0 \tag{58}$$

Constraints (57) and (58) relate several decision variables. Students often omit constraints of this type. As this problem shows, leaving out even one constraint may very well

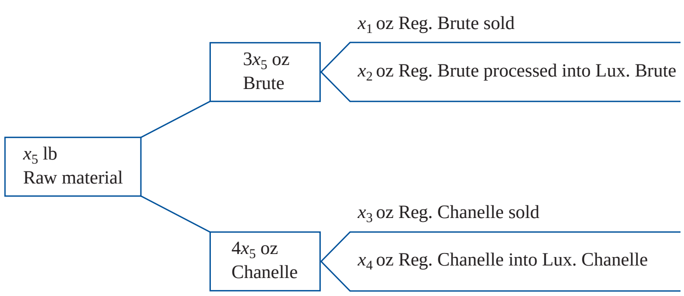


FIGURE 10
Production Process for Brute and Chanelle

lead to an unacceptable answer (such as an unbounded LP). If we combine (53)–(58) with the usual sign restrictions, we obtain the *correct* LP formulation.

$$\begin{aligned} \max z = 7x_1 + 14x_2 + 6x_3 + 10x_4 - 3x_5 & \\ \text{s.t.} \qquad x_5 &\le 4{,}000 \\ 3x_2 + 2x_4 + x_5 &\le 6{,}000 \\ x_1 + x_2 - 3x_5 &= 0 \\ x_3 + x_4 - 4x_5 &= 0 \\ x_i \ge 0 \quad (i = 1, 2, 3, 4, 5) & \end{aligned}$$

The optimal solution is $z = 172{,}666.667$, $x_1 = 11{,}333.333$ oz, $x_2 = 666.667$ oz, $x_3 = 16{,}000$ oz, $x_4 = 0$, and $x_5 = 4{,}000$ lb. Thus, Rylon should purchase all 4,000 lb of available raw material and produce 11,333.333 oz of Regular Brute, 666.667 oz of Luxury Brute, and 16,000 oz of Regular Chanelle. This production plan will contribute \$172,666.667 to Rylon's profits. In this problem, a fractional number of ounces seems reasonable, so the Divisibility Assumption holds.

We close our discussion of the Rylon problem by discussing an error that is made by many students. They reason that

$$1 \text{ lb raw material} = 3 \text{ oz Brute} + 4 \text{ oz Chanelle}$$

Because $x_1 + x_2 =$ total ounces of Brute produced, and $x_3 + x_4 =$ total ounces of Chanelle produced, students conclude that

$$x_5 = 3(x_1 + x_2) + 4(x_3 + x_4) \tag{59}$$

This equation might make sense as a statement for a computer program; in a sense, the variable x_5 is replaced by the right side of (59). As an LP constraint, however, (59) makes no sense. To see this, note that the left side has the units "pounds of raw material," and the term $3x_1$ on the right side has the units

$$\left(\frac{\text{Ounces of Brute}}{\text{Pounds of raw material}}\right)(\text{ounces of Brute})$$

Because some of the terms do not have the same units, (59) cannot be correct. *If there are doubts about a constraint, then make sure that all terms in the constraint have the same units.* This will avoid many formulation errors. (Of course, even if the units on both sides of a constraint are the same, the constraint may still be wrong.)

PROBLEMS

Group A

1 Sunco Oil has three different processes that can be used to manufacture various types of gasoline. Each process involves blending oils in the company's catalytic cracker. Running process 1 for an hour costs \$5 and requires 2 barrels of crude oil 1 and 3 barrels of crude oil 2. The output from running process 1 for an hour is 2 barrels of gas 1 and 1 barrel of gas 2. Running process 2 for an hour costs \$4 and requires 1 barrel of crude 1 and 3 barrels of crude 2. The output from running process 2 for an hour is 3 barrels of gas 2. Running process 3 for an hour costs \$1 and requires 2 barrels of crude 2 and 3 barrels of gas 2. The output from running process 3 for an hour is 2 barrels of gas 3. Each week, 200 barrels of crude 1, at \$2/barrel, and 300 barrels of crude 2, at \$3/barrel, may be purchased. All gas produced can be sold at the following per-barrel prices: gas 1, \$9; gas 2, \$10; gas 3, \$24. Formulate an LP whose solution will maximize revenues less costs. Assume that only 100 hours of time on the catalytic cracker are available each week.

2 Furnco manufactures tables and chairs. A table requires 40 board ft of wood, and a chair requires 30 board ft of wood. Wood may be purchased at a cost of $1 per board ft, and 40,000 board ft of wood are available for purchase. It takes 2 hours of skilled labor to manufacture an unfinished table or an unfinished chair. Three more hours of skilled labor will turn an unfinished table into a finished table, and 2 more hours of skilled labor will turn an unfinished chair into a finished chair. A total of 6,000 hours of skilled labor are available (and have already been paid for). All furniture produced can be sold at the following unit prices: unfinished table, $70; finished table, $140; unfinished chair, $60; finished chair, $110. Formulate an LP that will maximize the contribution to profit from manufacturing tables and chairs.

3 Suppose that in Example 11, 1 lb of raw material could be used to produce either 3 oz of Brute *or* 4 oz of Chanelle. How would this change the formulation?

4 Chemco produces three products: 1, 2, and 3. Each pound of raw material costs $25. It undergoes processing and yields 3 oz of product 1 and 1 oz of product 2. It costs $1 and takes 2 hours of labor to process each pound of raw material. Each ounce of product 1 can be used in one of three ways.

It can be sold for $10/oz.

It can be processed into 1 oz of product 2. This requires 2 hours of labor and costs $1.

It can be processed into 1 oz of product 3. This requires 3 hours of labor and costs $2.

Each ounce of product 2 can be used in one of two ways.

It can be sold for $20/oz.

It can be processed into 1 oz of product 3. This requires 1 hour of labor and costs $6.

Product 3 is sold for $30/oz. The maximum number of ounces of each product that can be sold is given in Table 23. A maximum of 25,000 hours of labor are available. Determine how Chemco can maximize profit.

TABLE 23

Product	Oz
1	5,000
2	5,000
3	3,000

Group B

5 A company produces A, B, and C and can sell these products in unlimited quantities at the following unit prices: A, $10; B, $56; C, $100. Producing a unit of A requires 1 hour of labor; a unit of B, 2 hours of labor plus 2 units of A; and a unit of C, 3 hours of labor plus 1 unit of B. Any A that is used to produce B cannot be sold. Similarly, any B that is used to produce C cannot be sold. A total of 40 hours of labor are available. Formulate an LP to maximize the company's revenues.

6 Daisy Drugs manufactures two drugs: 1 and 2. The drugs are produced by blending two chemicals: 1 and 2. By weight, drug 1 must contain at least 65% chemical 1, and drug 2 must contain at least 55% chemical 1. Drug 1 sells for $6/oz, and drug 2 sells for $4/oz. Chemicals 1 and 2 can be produced by one of two production processes. Running process 1 for an hour requires 3 oz of raw material and 2 hours skilled labor and yields 3 oz of each chemical. Running process 2 for an hour requires 2 oz of raw material and 3 hours of skilled labor and yields 3 oz of chemical 1 and 1 oz of chemical 2. A total of 120 hours of skilled labor and 100 oz of raw material are available. Formulate an LP that can be used to maximize Daisy's sales revenues.

7† Lizzie's Dairy produces cream cheese and cottage cheese. Milk and cream are blended to produce these two products. Both high-fat and low-fat milk can be used to produce cream cheese and cottage cheese. High-fat milk is 60% fat; low-fat milk is 30% fat. The milk used to produce cream cheese must average at least 50% fat and that for cottage cheese, at least 35% fat. At least 40% (by weight) of the inputs to cream cheese and at least 20% (by weight) of the inputs to cottage cheese must be cream. Both cottage cheese and cream cheese are produced by putting milk and cream through the cheese machine. It costs 40¢ to process 1 lb of inputs into a pound of cream cheese. It costs 40¢ to produce 1 lb of cottage cheese, but every pound of input for cottage cheese yields 0.9 lb of cottage cheese and 0.1 lb of waste. Cream can be produced by evaporating high-fat and low-fat milk. It costs 40¢ to evaporate 1 lb of high-fat milk. Each pound of high-fat milk that is evaporated yields 0.6 lb of cream. It costs 40¢ to evaporate 1 lb of low-fat milk. Each pound of low-fat milk that is evaporated yields 0.3 lb of cream. Each day, up to 3,000 lb of input may be sent through the cheese machine. Each day, at least 1,000 lb of cottage cheese and 1,000 lb of cream cheese must be produced. Up to 1,500 lb of cream cheese and 2,000 lb of cottage cheese can be sold each day. Cottage cheese is sold for $1.20/lb and cream cheese for $1.50/lb. High-fat milk is purchased for 80¢/lb and low-fat milk for 40¢/lb. The evaporator can process at most 2,000 lb of milk daily. Formulate an LP that can be used to maximize Lizzie's daily profit.

8 A company produces six products in the following fashion. Each unit of raw material purchased yields four units of product 1, two units of product 2, and one unit of product 3. Up to 1,200 units of product 1 can be sold, and up to 300 units of product 2 can be sold. Each unit of product 1 can be sold or processed further. Each unit of product 1 that is processed yields a unit of product 4. Demand for products 3 and 4 is unlimited. Each unit of product 2 can be sold or processed further. Each unit of product 2 that is processed further yields 0.8 unit of product 5 and 0.3 unit of product 6. Up to 1,000 units of product 5 can be sold, and up to 800 units of product 6 can be sold. Up to 3,000 units of raw material can be purchased at $6 per unit. Leftover units of products 5 and 6 must be destroyed. It costs $4 to destroy each leftover unit of product 5 and $3

†Based on Sullivan and Secrest (1985).

TABLE 24

Product	Sales Price ($)	Production Cost ($)
1	7	4
2	6	4
3	4	2
4	3	1
5	20	5
6	35	5

to destroy each leftover unit of product 6. Ignoring raw material purchase costs, the per-unit sales price and production costs for each product are shown in Table 24. Formulate an LP whose solution will yield a profit-maximizing production schedule.

9 Each week Chemco can purchase unlimited quantities of raw material at $6/lb. Each pound of purchased raw material can be used to produce either input 1 or input 2. Each pound of raw material can yield 2 oz of input 1, requiring 2 hours of processing time and incurring $2 in processing costs. Each pound of raw material can yield 3 oz of input 2, requiring 2 hours of processing time and incurring $4 in processing costs.

Two production processes are available. It takes 2 hours to run process 1, requiring 2 oz of input 1 and 1 oz of input 2. It costs $1 to run process 1. Each time process 1 is run 1 oz of product A and 1 oz of liquid waste are produced. Each time process 2 is run requires 3 hours of processing time, 2 oz of input 2 and 1 oz of input 1. Process 2 yields 1 oz of product B and .8 oz of liquid waste. Process 2 incurs $8 in costs.

Chemco can dispose of liquid waste in the Port Charles River or use the waste to produce product C or product D. Government regulations limit the amount of waste Chemco is allowed to dump into the river to 1,000 oz/week. One ounce of product C costs $4 to produce and sells for $11. One hour of processing time, 2 oz of input 1, and .8 oz of liquid waste are needed to produce an ounce of product C. One unit of product D costs $5 to produce and sells for $7. One hour of processing time, 2 oz of input 2, and 1.2 oz of liquid waste are needed to produce an ounce of product D.

At most 5,000 oz of product A and 5,000 oz of product B can be sold each week, but weekly demand for products C and D is unlimited. Product A sells for $18/oz and product B sells for $24/oz. Each week 6,000 hours of processing time is available. Formulate an LP whose solution will tell Chemco how to maximize weekly profit.

10 LIMECO owns a lime factory and sells six grades of lime (grades 1 through 6). The sales price per pound is given in Table 25. Lime is produced by kilns. If a kiln is run for an 8-hour shift, the amounts (in pounds) of each grade of lime given in Table 26 are produced. It costs $150 to run a kiln for an 8-hour shift. Each day the factory believes it can sell up to the amounts (in pounds) of lime given in Table 27.

Lime that is produced by the kiln may be reprocessed by using any one of the five processes described in Table 28.

For example, at a cost of $1/lb, a pound of grade 4 lime may be transformed into .5 lb of grade 5 lime and .5 lb of grade 6 lime.

Any extra lime leftover at the end of each day must be disposed of, with the disposal costs (per pound) given in Table 29.

Formulate an LP whose solution will tell LIMECO how to maximize their daily profit.

TABLE 25

Grade	1	2	3	4	5	6
Price($)	12	14	10	18	20	25

TABLE 26

Grade	1	2	3	4	5	6
Amount produced	2	3	1	1.5	2	3

TABLE 27

Grade	1	2	3	4	5	6
Maximum demand	20	30	40	35	25	50

11 Chemco produces three products: A, B, and C. They can sell up to 30 pounds of each product at the following prices (per pound): product A, $10; product B, $12; product C, $20. Chemco purchases raw material at $5/lb. Each pound of raw material can be used to produce *either* 1 lb of A or 1 lb of B. For a cost of $3/lb processed, product A can be converted to .6 lb of product B and .4 lb of product C. For a cost of $2/lb processed, product B can be converted to .8 lb of product C. Formulate an LP whose solution will tell Chemco how to maximize their profit.

12 Chemco produces 3 chemicals: B, C, and D. They begin by purchasing chemical A for a cost of $6/100 liters. For an

TABLE 28

Input (1 Lb)	Output	Cost ($ per Lb of Input)
Grade 1	.3 lb Grade 3	
	.2 lb Grade 4	2
	.3 lb Grade 5	
	.2 lb Grade 6	
Grade 2	1 lb Grade 6	1
Grade 3	.8 lb Grade 4	1
Grade 4	.5 lb Grade 5	1
	.5 lb Grade 6	
Grade 5	.9 lb Grade 6	2

TABLE 29

Grade	1	2	3	4	5	6
Cost of Disposition ($)	3	2	3	2	4	2

additional cost of $3 and the use of 3 hours of skilled labor, 100 liters of A can be transformed into 40 liters of C and 60 liters of B. Chemical C can either be sold or processed further. It costs $1 and takes 1 hour of skilled labor to process 100 liters of C into 60 liters of D and 40 liters of B. For each chemical the sales price per 100 liters and the maximum amount (in 100s of liters) that can be sold are given in Table 30.

A maximum of 200 labor hours are available. Formulate an LP whose solution will tell Chemco how to maximize their profit.

13 Carrington Oil produces two types of gasoline, gas 1 and gas 2, from two types of crude oil, crude 1 and crude 2. Gas 1 is allowed to contain up to 4% impurities, and gas 2 is allowed to contain up to 3% impurities. Gas 1 sells for $8 per barrel, whereas gas 2 sells for $12 per barrel. Up to 4,200 barrels of gas 1 and up to 4,300 barrels of gas 2 can be sold. The cost per barrel of each crude, availability, and the level of impurities in each crude are as shown in Table 31. Before blending the crude oil into gas, any amount of each crude can be "purified" for a cost of $0.50 per barrel. Purification eliminates half the impurities in the crude oil. Determine how to maximize profit.

14 You have been put in charge of the Melrose oil refinery. The refinery produces gas and heating oil from crude oil. Gas sells for $8 per barrel and must have an average "grade level" of at least 9. Heating oil sells for $6 a barrel and must have an average grade level of at least 7. At most, 2,000 barrels of gas and 600 barrels of heating oil can be sold. Incoming crude can be processed by one of three methods. The per barrel yield and per barrel cost of each processing method are shown in Table 32. For example, if we refine 1 barrel of incoming crude by method 1, it costs us $3.40 and yields .2 barrels of grade 6, .2 barrels of grade 8, and .6 barrels of grade 10.

Before being processed into gas and heating oil, processed grades 6 and 8 may be sent through the catalytic cracker to improve their quality. For $1.30 per barrel, a barrel of grade 6 may be "cracked" into a barrel of grade 8. For $2 per barrel, a barrel of grade 8 may be cracked into a barrel of grade 10. Any leftover processed or cracked oil that cannot be used for heating oil or gas must be disposed of at a cost of $0.20 per barrel. Determine how to maximize the refinery's profit.

TABLE 30

	B	C	D
Price ($)	12	16	26
Maximum demand	30	60	40

TABLE 31

Oil	Cost per Barrel ($)	Impurity Level (%)	Availability (Barrels)
Crude 1	6	10%	5,000
Crude 2	8	2%	4,500

TABLE 32

Method	Grade 6	Grade 8	Grade 10	Cost ($)
1	.2	.2	.6	3.40
2	.3	.3	.4	3.00
3	.4	.4	.2	2.60

3.10 Using Linear Programming to Solve Multiperiod Decision Problems: An Inventory Model

Up to this point, all the LP formulations we have discussed are examples of *static,* or *one-period, models.* In a static model, we assume that all decisions are made at a single point in time. The rest of the examples in this chapter show how linear programming can be used to determine optimal decisions in **multiperiod,** or **dynamic, models.** Dynamic models arise when the decision maker makes decisions at more than one point in time. In a dynamic model, decisions made during the current period influence decisions made during future periods. For example, consider a company that must determine how many units of a product should be produced during each month. If it produced a large number of units during the current month, this would reduce the number of units that should be produced during future months. The examples discussed in Sections 3.10–3.12 illustrate how earlier decisions affect later decisions. We will return to dynamic decision models when we study dynamic programming in Chapters 18 and 19.

EXAMPLE 14 **Sailco Inventory**

Sailco Corporation must determine how many sailboats should be produced during each of the next four quarters (one quarter = three months). The demand during each of the next four quarters is as follows: first quarter, 40 sailboats; second quarter, 60 sailboats; third quarter, 75 sailboats; fourth quarter, 25 sailboats. Sailco must meet demands on time. At the beginning of the first quarter, Sailco has an inventory of 10 sailboats. At the beginning of each quarter, Sailco must decide how many sailboats should be produced during that quarter. For simplicity, we assume that sailboats manufactured during a quarter can be used to meet demand for that quarter. During each quarter, Sailco can produce up to 40 sailboats with regular-time labor at a total cost of $400 per sailboat. By having employees work overtime during a quarter, Sailco can produce additional sailboats with overtime labor at a total cost of $450 per sailboat.

At the end of each quarter (after production has occurred and the current quarter's demand has been satisfied), a carrying or holding cost of $20 per sailboat is incurred. Use linear programming to determine a production schedule to minimize the sum of production and inventory costs during the next four quarters.

Solution For each quarter, Sailco must determine the number of sailboats that should be produced by regular-time and by overtime labor. Thus, we define the following decision variables:

x_t = number of sailboats produced by regular-time labor (at $400/boat) during quarter t $\quad (t = 1, 2, 3, 4)$

y_t = number of sailboats produced by overtime labor (at $450/boat) during quarter t $\quad (t = 1, 2, 3, 4)$

It is convenient to define decision variables for the inventory (number of sailboats on hand) at the end of each quarter:

i_t = number of sailboats on hand at end of quarter t $\quad (t = 1, 2, 3, 4)$

Sailco's total cost may be determined from

$$\begin{aligned}\text{Total cost} &= \text{cost of producing regular-time boats}\\ &\quad + \text{cost of producing overtime boats} + \text{inventory costs}\\ &= 400(x_1 + x_2 + x_3 + x_4) + 450(y_1 + y_2 + y_3 + y_4)\\ &\quad + 20(i_1 + i_2 + i_3 + i_4)\end{aligned}$$

Thus, Sailco's objective function is

$$\begin{aligned}\min z = {} & 400x_1 + 400x_2 + 400x_3 + 400x_4 + 450y_1 + 450y_2\\ & + 450y_3 + 450y_4 + 20i_1 + 20i_2 + 20i_3 + 20i_4 \end{aligned} \tag{60}$$

Before determining Sailco's constraints, we make two observations that will aid in formulating multiperiod production-scheduling models.

For quarter t,

$$\begin{aligned}\text{Inventory at end of quarter } t &= \text{inventory at end of quarter } (t - 1)\\ &\quad + \text{quarter } t \text{ production} - \text{quarter } t \text{ demand}\end{aligned}$$

This relation plays a key role in formulating almost all multiperiod production-scheduling models. If we let d_t be the demand during period t (thus, $d_1 = 40$, $d_2 = 60$, $d_3 = 75$, and $d_4 = 25$), our observation may be expressed in the following compact form:

$$i_t = i_{t-1} + (x_t + y_t) - d_t \qquad (t = 1, 2, 3, 4) \tag{61}$$

In (61), i_0 = inventory at end of quarter 0 = inventory at beginning of quarter 1 = 10. For example, if we had 20 sailboats on hand at the end of quarter 2 ($i_2 = 20$) and produced 65 sailboats during quarter 3 (this means $x_3 + y_3 = 65$), what would be our ending third-quarter inventory? Simply the number of sailboats on hand at the end of quarter 2 plus the sailboats produced during quarter 3, less quarter 3's demand of 75. In this case, $i_3 = 20 + 65 - 75 = 10$, which agrees with (61). Equation (61) relates decision variables associated with different time periods. In formulating any multiperiod LP model, the hardest step is usually finding the relation (such as (61)) that relates decision variables from different periods.

We also note that quarter t's demand will be met on time if and only if (sometimes written *iff*) $i_t \geq 0$. To see this, observe that $i_{t-1} + (x_t + y_t)$ is available to meet period t's demand, so that period t's demand will be met if and only if

$$i_{t-1} + (x_t + y_t) \geq d_t \qquad \text{or} \qquad i_t = i_{t-1} + (x_t + y_t) - d_t \geq 0$$

This means that the sign restrictions $i_t \geq 0$ ($t = 1, 2, 3, 4$) will ensure that each quarter's demand will be met on time.

We can now determine Sailco's constraints. First, we use the following four constraints to ensure that each period's regular-time production will not exceed 40: $x_1, x_2, x_3, x_4 \leq 40$. Then we add constraints of the form (61) for each time period ($t = 1, 2, 3, 4$). This yields the following four constraints:

$$\begin{aligned} i_1 &= 10 + x_1 + y_1 - 40 \qquad & i_2 &= i_1 + x_2 + y_2 - 60 \\ i_3 &= i_2 + x_3 + y_3 - 75 \qquad & i_4 &= i_3 + x_4 + y_4 - 25 \end{aligned}$$

Adding the sign restrictions $x_t \geq 0$ (to rule out negative production levels) and $i_t \geq 0$ (to ensure that each period's demand is met on time) yields the following formulation:

$$\begin{aligned} \min z = {} & 400x_1 + 400x_2 + 400x_3 + 400x_4 + 450y_1 + 450y_2 + 450y_3 + 450y_4 \\ & + 20i_1 + 20i_2 + 20i_3 + 20i_4 \\ \text{s.t.} \quad & x_1 \leq 40, \quad x_2 \leq 40, \quad x_3 \leq 40, \quad x_4 \leq 40 \\ & i_1 = 10 + x_1 + y_1 - 40, \quad i_2 = i_1 + x_2 + y_2 - 60 \\ & i_3 = i_2 + x_3 + y_3 - 75, \quad i_4 = i_3 + x_4 + y_4 - 25 \\ & i_t \geq 0, \quad y_t \geq 0, \quad \text{and} \quad x_t \geq 0 \quad (t = 1, 2, 3, 4) \end{aligned}$$

The optimal solution to this problem is $z = 78{,}450$; $x_1 = x_2 = x_3 = 40$; $x_4 = 25$; $y_1 = 0$; $y_2 = 10$; $y_3 = 35$; $y_4 = 0$; $i_1 = 10$; $i_2 = i_3 = i_4 = 0$. Thus, the minimum total cost that Sailco can incur is $78,450. To incur this cost, Sailco should produce 40 sailboats with regular-time labor during quarters 1–3 and 25 sailboats with regular-time labor during quarter 4. Sailco should also produce 10 sailboats with overtime labor during quarter 2 and 35 sailboats with overtime labor during quarter 3. Inventory costs will be incurred only during quarter 1.

Some readers might worry that our formulation allows Sailco to use overtime production during quarter t even if period t's regular production is less than 40. True, our formulation does not make such a schedule infeasible, but any production plan that had $y_t > 0$ and $x_t < 40$ could not be optimal. For example, consider the following two production schedules:

$$\begin{aligned} \text{Production schedule A} &= x_1 = x_2 = x_3 = 40; \quad x_4 = 25; \\ & \quad y_2 = 10; \quad y_3 = 25; \quad y_4 = 0 \\ \text{Production schedule B} &= x_1 = 40; \quad x_2 = 30; \quad x_3 = 30; \quad x_4 = 25; \\ & \quad y_2 = 20; \quad y_3 = 35; \quad y_4 = 0 \end{aligned}$$

Schedules A and B both have the same production level during each period. This means that both schedules will have identical inventory costs. Also, both schedules are feasible, but schedule B incurs more overtime costs than schedule A. Thus, in minimizing costs, schedule B (or any schedule having $y_t > 0$ and $x_t < 40$) would never be chosen.

In reality, an LP such as Example 14 would be implemented by using a **rolling horizon,** which works in the following fashion. After solving Example 14, Sailco would implement only the quarter 1 production strategy (produce 40 boats with regular-time labor). Then the company would observe quarter 1's actual demand. Suppose quarter 1's actual demand is 35 boats. Then quarter 2 begins with an inventory of $10 + 40 - 35 = 15$ boats. We now make a forecast for quarter 5 demand (suppose the forecast is 36). Next determine production for quarter 2 by solving an LP in which quarter 2 is the first quarter, quarter 5 is the final quarter, and beginning inventory is 15 boats. Then quarter 2's production would be determined by solving the following LP:

$$\begin{aligned}
\min z &= 400(x_2 + x_3 + x_4 + x_5) + 450(y_2 + y_3 + y_4 + y_5) + 20(i_2 + i_3 + i_4 + i_5)\\
\text{s.t.}\quad & x_2 \le 40, \quad x_3 \le 40, \quad x_4 \le 40, \quad x_5 \le 40\\
& i_2 = 15 + x_2 + y_2 - 60, \quad i_3 = i_2 + x_3 + y_3 - 75\\
& i_4 = i_3 + x_4 + y_4 - 25, \quad i_5 = i_4 + x_5 + y_5 - 36\\
& i_t \ge 0, \quad y_t \ge 0, \quad \text{and} \quad x_t \ge 0 \quad (t = 2, 3, 4, 5)
\end{aligned}$$

Here, x_5 = quarter 5's regular-time production, y_5 = quarter 5's overtime production, and i_5 = quarter 5's ending inventory. The optimal values of x_2 and y_2 for this LP are then used to determine quarter 2's production. Thus, each quarter, an LP (with a planning horizon of four quarters) is solved to determine the current quarter's production. Then current demand is observed, demand is forecasted for the next four quarters, and the process repeats itself. This technique of "rolling planning horizon" is the method by which most dynamic or multiperiod LP models are implemented in real-world applications.

Our formulation of the Sailco problem has several other limitations.

1 Production cost may not be a linear function of the quantity produced. This would violate the Proportionality Assumption. We discuss how to deal with this problem in Chapters 9 and 13.

2 Future demands may not be known with certainty. In this situation, the Certainty Assumption is violated.

3 We have required Sailco to meet all demands on time. Often companies can meet demands during later periods but are assessed a penalty cost for demands that are not met on time. For example, if demand is not met on time, then customer displeasure may result in a loss of future revenues. If demand can be met during later periods, then we say that demands can be **backlogged.** Our current LP formulation can be modified to incorporate backlogging (see Problem 1 of Section 4.12).

4 We have ignored the fact that quarter-to-quarter variations in the quantity produced may result in extra costs (called **production-smoothing costs.**) For example, if we increase production a great deal from one quarter to the next, this will probably require the costly training of new workers. On the other hand, if production is greatly decreased from one quarter to the next, extra costs resulting from laying off workers may be incurred. In Section 4.12, we modify the present model to account for smoothing costs.

5 If any sailboats are left at the end of the last quarter, we have assigned them a value of zero. This is clearly unrealistic. In any inventory model with a finite horizon, the inventory left at the end of the last period should be assigned a **salvage value** that is indicative of the worth of the final period's inventory. For example, if Sailco feels that each sailboat left at the end of quarter 4 is worth \$400, then a term $-400i_4$ (measuring the worth of quarter 4's inventory) should be added to the objective function.

PROBLEMS

Group A

1 A customer requires during the next four months, respectively, 50, 65, 100, and 70 units of a commodity (no backlogging is allowed). Production costs are \$5, \$8, \$4, and \$7 per unit during these months. The storage cost from one month to the next is \$2 per unit (assessed on ending inventory). It is estimated that each unit on hand at the end of month 4 could be sold for \$6. Formulate an LP that will minimize the net cost incurred in meeting the demands of the next four months.

2 A company faces the following demands during the next three periods: period 1, 20 units; period 2, 10 units; period 3, 15 units. The unit production cost during each period is as follows: period 1—\$13; period 2—\$14; period 3—\$15. A holding cost of \$2 per unit is assessed against each period's ending inventory. At the beginning of period 1, the company has 5 units on hand.

In reality, not all goods produced during a month can be used to meet the current month's demand. To model this fact, we assume that only one half of the goods produced during a period can be used to meet the current period's demands. Formulate an LP to minimize the cost of meeting the demand for the next three periods. (*Hint:* Constraints such as $i_1 = x_1 + 5 - 20$ are certainly needed. Unlike our example, however, the constraint $i_1 \geq 0$ will not ensure that period 1's demand is met. For example, if $x_1 = 20$, then $i_1 \geq 0$ will hold, but because only $\frac{1}{2}(20) = 10$ units of period 1 production can be used to meet period 1's demand, $x_1 = 20$ would not be feasible. Try to think of a type of constraint that will ensure that what is available to meet each period's demand is at least as large as that period's demand.)

Group B

3 James Beerd bakes cheesecakes and Black Forest cakes. During any month, he can bake at most 65 cakes. The costs per cake and the demands for cakes, which must be met on time, are listed in Table 33. It costs 50¢ to hold a cheesecake, and 40¢ to hold a Black Forest cake, in inventory for a month. Formulate an LP to minimize the total cost of meeting the next three months' demands.

4 A manufacturing company produces two types of products: A and B. The company has agreed to deliver the products on the schedule shown in Table 34. The company has two assembly lines, 1 and 2, with the available production hours shown in Table 35. The production rates for each assembly line and product combination, in terms of hours per product, are shown in Table 36. It takes 0.15 hour to manufacture 1 unit of product A on line 1, and so on. It costs \$5 per hour of line time to produce any product. The inventory carrying cost per month for each product is 20¢ per unit (charged on each month's ending inventory). Currently, there are 500 units of A and 750 units of B in inventory. Management would like at least 1,000 units of each product in inventory at the end of April. Formulate an LP to determine the production schedule that minimizes the total cost incurred in meeting demands on time.

5 During the next two months, General Cars must meet (on time) the following demands for trucks and cars: month 1—400 trucks, 800 cars; month 2—300 trucks, 300 cars. During each month, at most 1,000 vehicles can be produced. Each truck uses 2 tons of steel, and each car uses 1 ton of steel. During month 1, steel costs \$400 per ton; during month 2, steel costs \$600 per ton. At most, 1,500 tons of steel may be purchased each month (steel may only be used

TABLE 34

Date	A	B
March 31	5,000	2,000
April 30	8,000	4,000

TABLE 35

	Production Hours Available	
Month	Line 1	Line 2
March	800	2,000
April	400	1,200

TABLE 36

	Production Rate	
Product	Line 1	Line 2
A	0.15	0.16
B	0.12	0.14

TABLE 33

	Month 1		Month 2		Month 3	
Item	Demand	Cost/Cake (\$)	Demand	Cost/Cake (\$)	Demand	Cost/Cake (\$)
Cheesecake	40	3.00	30	3.40	20	3.80
Black Forest	20	2.50	30	2.80	10	3.40

during the month in which it is purchased). At the beginning of month 1, 100 trucks and 200 cars are in inventory. At the end of each month, a holding cost of $150 per vehicle is assessed. Each car gets 20 mpg, and each truck gets 10 mpg. During each month, the vehicles produced by the company must average at least 16 mpg. Formulate an LP to meet the demand and mileage requirements at minimum cost (include steel costs and holding costs).

6 Gandhi Clothing Company produces shirts and pants. Each shirt requires 2 sq yd of cloth, each pair of pants, 3. During the next two months, the following demands for shirts and pants must be met (on time): month 1—10 shirts, 15 pairs of pants; month 2—12 shirts, 14 pairs of pants. During each month, the following resources are available: month 1—90 sq yd of cloth; month 2—60 sq yd. (Cloth that is available during month 1 may, if unused during month 1, be used during month 2.)

During each month, it costs $4 to make an article of clothing with regular-time labor and $8 with overtime labor. During each month, a total of at most 25 articles of clothing may be produced with regular-time labor, and an unlimited number of articles of clothing may be produced with overtime labor. At the end of each month, a holding cost of $3 per article of clothing is assessed. Formulate an LP that can be used to meet demands for the next two months (on time) at minimum cost. Assume that at the beginning of month 1, 1 shirt and 2 pairs of pants are available.

7 Each year, Paynothing Shoes faces demands (which must be met on time) for pairs of shoes as shown in Table 37. Workers work three consecutive quarters and then receive one quarter off. For example, a worker may work during quarters 3 and 4 of one year and quarter 1 of the next year. During a quarter in which a worker works, he or she can produce up to 50 pairs of shoes. Each worker is paid $500 per quarter. At the end of each quarter, a holding cost of $50 per pair of shoes is assessed. Formulate an LP that can be used to minimize the cost per year (labor + holding) of meeting the demands for shoes. To simplify matters, assume that at the end of each year, the ending inventory is zero. (*Hint:* It is allowable to assume that a given worker will get the same quarter off during each year.)

TABLE 37

Quarter 1	Quarter 2	Quarter 3	Quarter 4
600	300	800	100

8 A company must meet (on time) the following demands: quarter 1—30 units; quarter 2—20 units; quarter 3—40 units. Each quarter, up to 27 units can be produced with regular-time labor, at a cost of $40 per unit. During each quarter, an unlimited number of units can be produced with overtime labor, at a cost of $60 per unit. Of all units produced, 20% are unsuitable and cannot be used to meet demand. Also, at the end of each quarter, 10% of all units on hand spoil and cannot be used to meet any future demands. After each quarter's demand is satisfied and spoilage is accounted for, a cost of $15 per unit is assessed against the quarter's ending inventory. Formulate an LP that can be used to minimize the total cost of meeting the next three quarters' demands. Assume that 20 usable units are available at the beginning of quarter 1.

9 Donovan Enterprises produces electric mixers. During the next four quarters, the following demands for mixers must be met on time: quarter 1—4,000; quarter 2—2,000; quarter 3—3,000; quarter 4—10,000. Each of Donovan's workers works three quarters of the year and gets one quarter off. Thus, a worker may work during quarters 1, 2, and 4 and get quarter 3 off. Each worker is paid $30,000 per year and (if working) can produce up to 500 mixers during a quarter. At the end of each quarter, Donovan incurs a holding cost of $30 per mixer on each mixer in inventory. Formulate an LP to help Donovan minimize the cost (labor and inventory) of meeting the next year's demand (on time). At the beginning of quarter 1, 600 mixers are available.

3.11 Multiperiod Financial Models

The following example illustrates how linear programming can be used to model multiperiod cash management problems. The key is to determine the relations of cash on hand during different periods.

EXAMPLE 15 Finco Multiperiod Investment

Finco Investment Corporation must determine investment strategy for the firm during the next three years. Currently (time 0), $100,000 is available for investment. Investments A, B, C, D, and E are available. The cash flow associated with investing $1 in each investment is given in Table 38.

For example, $1 invested in investment B requires a $1 cash outflow at time 1 and returns 50¢ at time 2 and $1 at time 3. To ensure that the company's portfolio is diversified, Finco requires that at most $75,000 be placed in any single investment. In addition to investments A–E, Finco can earn interest at 8% per year by keeping uninvested cash in

TABLE 38

	Cash Flow ($) at Time*			
	0	1	2	3
A	−1	+0.50	+1	0
B	0	−1	+0.50	+1
C	−1	+1.2	0	0
D	−1	0	0	+1.9
E	0	0	−1	+1.5

*Note: Time 0 = present; time 1 = 1 year from now; time 2 = 2 years from now; time 3 = 3 years from now.

money market funds. Returns from investments may be immediately reinvested. For example, the positive cash flow received from investment C at time 1 may immediately be reinvested in investment B. Finco cannot borrow funds, so the cash available for investment at any time is limited to cash on hand. Formulate an LP that will maximize cash on hand at time 3.

Solution Finco must decide how much money should be placed in each investment (including money market funds). Thus, we define the following decision variables:

A = dollars invested in investment A

B = dollars invested in investment B

C = dollars invested in investment C

D = dollars invested in investment D

E = dollars invested in investment E

S_t = dollars invested in money market funds at time t $\quad (t = 0, 1, 2)$

Finco wants to maximize cash on hand at time 3. At time 3, Finco's cash on hand will be the sum of all cash inflows at time 3. From the description of investments A–E and the fact that from time 2 to time 3, S_2 will increase to $1.08S_2$,

$$\text{Time 3 cash on hand} = B + 1.9D + 1.5E + 1.08S_2$$

Thus, Finco's objective function is

$$\max z = B + 1.9D + 1.5E + 1.08S_2 \tag{62}$$

In multiperiod financial models, the following type of constraint is usually used to relate decision variables from different periods:

Cash available at time t = cash invested at time t
+ uninvested cash at time t that is carried over to time $t + 1$

If we classify money market funds as investments, we see that

$$\text{Cash available at time } t = \text{cash invested at time } t \tag{63}$$

Because investments A, C, D, and S_0 are available at time 0, and $100,000 is available at time 0, (63) for time 0 becomes

$$100{,}000 = A + C + D + S_0 \tag{64}$$

At time 1, $0.5A + 1.2C + 1.08S_0$ is available for investment, and investments B and S_1 are available. Then for $t = 1$, (63) becomes

$$0.5A + 1.2C + 1.08S_0 = B + S_1 \quad (65)$$

At time 2, $A + 0.5B + 1.08S_1$ is available for investment, and investments E and S_2 are available. Thus, for $t = 2$, (63) reduces to

$$A + 0.5B + 1.08S_1 = E + S_2 \quad (66)$$

Let's not forget that at most $75,000 can be placed in any of investments A–E. To take care of this, we add the constraints

$$A \le 75{,}000 \quad (67)$$
$$B \le 75{,}000 \quad (68)$$
$$C \le 75{,}000 \quad (69)$$
$$D \le 75{,}000 \quad (70)$$
$$E \le 75{,}000 \quad (71)$$

Combining (62) and (64)–(71) with the sign restrictions (all variables ≥ 0) yields the following LP:

$$\begin{aligned}
\max z &= B + 1.9D + 1.5E + 1.08S_2 \\
\text{s.t.} \quad A + C + D + S_0 &= 100{,}000 \\
0.5A + 1.2C + 1.08S_0 &= B + S_1 \\
A + 0.5B + 1.08S_1 &= E + S_2 \\
A &\le 75{,}000 \\
B &\le 75{,}000 \\
C &\le 75{,}000 \\
D &\le 75{,}000 \\
E &\le 75{,}000 \\
A, B, C, D, E, S_0, S_1, S_2 &\ge 0
\end{aligned}$$

We find the optimal solution to be $z = 218{,}500$, $A = 60{,}000$, $B = 30{,}000$, $D = 40{,}000$, $E = 75{,}000$, $C = S_0 = S_1 = S_2 = 0$. Thus, Finco should not invest in money market funds. At time 0, Finco should invest $60,000 in A and $40,000 in D. Then, at time 1, the $30,000 cash inflow from A should be invested in B. Finally, at time 2, the $60,000 cash inflow from A and the $15,000 cash inflow from B should be invested in E. At time 3, Finco's $100,000 will have grown to $218,500.

You might wonder how our formulation ensures that Finco never invests more money at any time than the firm has available. This is ensured by the fact that each variable S_i must be nonnegative. For example, $S_0 \ge 0$ is equivalent to $100{,}000 - A - C - D \ge 0$, which ensures that at most $100,000 will be invested at time 0.

Real-World Application

Using LP to Optimize Bond Portfolios

Many Wall Street firms buy and sell bonds. Rohn (1987) discusses a bond selection model that maximizes profit from bond purchases and sales subject to constraints that minimize the firm's risk exposure. See Problem 4 for a simplified version of this model.

PROBLEMS

Group A

1 A consultant to Finco claims that Finco's cash on hand at time 3 is the sum of the cash inflows from all investments, not just those investments yielding a cash inflow at time 3. Thus, the consultant claims that Finco's objective function should be

$$\max z = 1.5A + 1.5B + 1.2C + 1.9D + 1.5E + 1.08S_0 + 1.08S_1 + 1.08S_2$$

Explain why the consultant is incorrect.

2 Show that Finco's objective function may also be written as

$$\max z = 100{,}000 + 0.5A + 0.5B + 0.2C + 0.9D + 0.5E + 0.08S_0 + 0.08S_1 + 0.08S_2$$

3 At time 0, we have $10,000. Investments A and B are available; their cash flows are shown in Table 39. Assume that any money not invested in A or B earns *no* interest. Formulate an LP that will maximize cash on hand at time 3. Can you guess the optimal solution to this problem?

Group B

4[†] Broker Steve Johnson is currently trying to maximize his profit in the bond market. Four bonds are available for purchase and sale, with the bid and ask price of each bond as shown in Table 40. Steve can buy up to 1,000 units of each bond at the ask price or sell up to 1,000 units of each bond at the bid price. During each of the next three years, the person who sells a bond will pay the owner of the bond the cash payments shown in Table 41.

Steve's goal is to maximize his revenue from selling bonds less his payment for buying bonds, subject to the constraint that after each year's payments are received, his current cash position (due only to cash payments from bonds and not purchases or sale of bonds) is nonnegative. Assume that cash payments are discounted, with a payment of $1 one year from now being equivalent to a payment of 90¢ now. Formulate an LP to maximize net profit from buying and selling bonds, subject to the arbitrage constraints previously described. Why do you think we limit the number of units of each bond that can be bought or sold?

TABLE 39

Time	A	B
0	−$1	$0
1	$0.2	−$1
2	$1.5	$0
3	$0	$1.0

TABLE 40

Bond	Bid Price	Ask Price
1	980	990
2	970	985
3	960	972
4	940	954

[†]Based on Rohn (1987).

TABLE 41

Year	Bond 1	Bond 2	Bond 3	Bond 4
1	100	80	70	60
2	110	90	80	50
3	1,100	1,120	1,090	1,110

TABLE 42

Month	Cash Flow	Month	Cash Flow
January	−12	July	−7
February	−10	August	−2
March	−8	September	15
April	−10	October	12
May	−4	November	−7
June	5	December	45

5 A small toy store, Toyco projects the monthly cash flows (in thousands of dollars) in Table 42 during the year 2003. A negative cash flow means that cash outflows exceed cash inflows to the business. To pay its bills, Toyco will need to borrow money early in the year. Money can be borrowed in two ways:

a Taking out a long-term one-year loan in January. Interest of 1% is charged each month, and the loan must be paid back at the end of December.

b Each month money can be borrowed from a short-term bank line of credit. Here, a monthly interest rate of 1.5% is charged. All short-term loans must be paid off at the end of December.

At the end of each month, excess cash earns 0.4% interest. Formulate an LP whose solution will help Toyco maximize its cash position at the beginning of January, 2004.

6 Consider Problem 5 with the following modification: Each month Toyco can delay payments on some or all of the cash owed for the current month. This is called "stretching payments." Payments may be stretched for only one month, and a 1% penalty is charged on the amount stretched. Thus, if it stretches payments on $10,000 cash owed in January, then it must pay 10,000(1.01) = $10,100 in February. With this modification, formulate an LP that would help Toyco maximize its cash on hand at the beginning of January 1, 2004.

7 Suppose we are borrowing \$1,000 at 12% annual interest with 60 monthly payments. Assume equal payments are made at the end of month 1, month 2, . . . month 60. We know that entering into Excel the function

$$= \text{PMT}(.01, 60, 1{,}000)$$

would yield the monthly payment (\$22.24).

It is instructive to use LP to determine the montly payment. Let p be the (unknown) monthly payment. Each month we owe .01 · (our current unpaid balance) in interest. The remainder of our monthly payment is used to reduce the unpaid balance. For example, suppose we paid \$30 each month. At the beginning of month 1, our unpaid balance is \$1,000. Of our month 1 payment, \$10 goes to interest and \$20 to paying off the unpaid balance. Then we would begin month 2 with an unpaid balance of \$980. The trick is to use LP to determine the monthly payment that will pay off the loan at the end of month 60.

8 You are a CFA (chartered financial analyst). Madonna has come to you because she needs help paying off her credit card bills. She owes the amounts on her credit cards shown in Table 43. Madonna is willing to allocate up to \$5,000 per month to pay off these credit cards. All cards must be paid off within 36 months. Madonna's goal is to minimize the total of all her payments. To solve this problem, you must understand how interest on a loan works. To illustrate, suppose Madonna pays \$5,000 on Saks during month 1. Then her Saks balance at the beginning of month 2 is

$$20{,}000 - (5{,}000 - .005(20{,}000))$$

This follows because during month 1 Madonna incurs .005(20,000) in interest charges on her Saks card. Help Madonna solve her problems!

9 Winstonco is considering investing in three projects. If we fully invest in a project, the realized cash flows (in millions of dollars) will be as shown in Table 44. For example, project 1 requires cash outflow of \$3 million today and returns \$5.5 million 3 years from now. Today we have \$2 million in cash. At each time point (0, .5, 1, 1.5, 2, and 2.5 years from today) we may, if desired, borrow up to \$2 million at 3.5% (per 6 months) interest. Leftover cash earns 3% (per 6 months) interest. For example, if after borrowing and investing at time 0 we have \$1 million we would receive \$30,000 in interest at time .5 years. Winstonco's goal is to maximize cash on hand after it accounts for time 3 cash flows. What investment and borrowing strategy should be used? Remember that we may invest in a fraction of a project. For example, if we invest in .5 of project 3, then we have cash outflows of −\$1 million at time 0 and .5.

TABLE 43

Card	Balance (\$)	Monthly Rate (%)
Saks Fifth Avenue	20,000	.5
Bloomingdale's	50,000	1
Macys	40,000	1.5

TABLE 44

	Cash Flow		
Time (Years)	Project 1	Project 2	Project 3
0	−3	−2	−2
.5	−1	−.5	−2
1	+1.8	1.5	−1.8
1.5	1.4	1.5	1
2	1.8	1.5	1
2.5	1.8	.2	1
3	5.5	−1	6

3.12 Multiperiod Work Scheduling

In Section 3.5, we saw that linear programming could be used to schedule employees in a static environment where demand did not change over time. The following example (a modified version of a problem from Wagner [1975]) shows how LP can be used to schedule employee training when a firm faces demand that changes over time.

EXAMPLE 16 Multiperiod Work Scheduling

CSL is a chain of computer service stores. The number of hours of skilled repair time that CSL requires during the next five months is as follows:

Month 1 (January): 6,000 hours

Month 2 (February): 7,000 hours

Month 3 (March): 8,000 hours

Month 4 (April): 9,500 hours

Month 5 (May): 11,000 hours

At the beginning of January, 50 skilled technicians work for CSL. Each skilled technician can work up to 160 hours per month. To meet future demands, new technicians must be trained. It takes one month to train a new technician. During the month of training, a trainee must be supervised for 50 hours by an experienced technician. Each experienced technician is paid \$2,000 a month (even if he or she does not work the full 160 hours). During the month of training, a trainee is paid \$1,000 a month. At the end of each month, 5% of CSL's experienced technicians quit to join Plum Computers. Formulate an LP whose solution will enable CSL to minimize the labor cost incurred in meeting the service requirements for the next five months.

Solution CSL must determine the number of technicians who should be trained during month t ($t = 1, 2, 3, 4, 5$). Thus, we define

$$x_t = \text{number of technicians trained during month } t \qquad (t = 1, 2, 3, 4, 5)$$

CSL wants to minimize total labor cost during the next five months. Note that

$$\text{Total labor cost} = \text{cost of paying trainees} + \text{cost of paying experienced technicians}$$

To express the cost of paying experienced technicians, we need to define, for $t = 1, 2, 3, 4, 5$,

$$y_t = \text{number of experienced technicians at the beginning of month } t$$

Then

$$\begin{aligned}\text{Total labor cost} = {} & (1{,}000x_1 + 1{,}000x_2 + 1{,}000x_3 + 1{,}000x_4 + 1{,}000x_5) \\ & + (2{,}000y_1 + 2000y_2 + 2{,}000y_3 + 2{,}000y_4 + 2{,}000y_5)\end{aligned}$$

Thus, CSL's objective function is

$$\begin{aligned}\min z = {} & 1{,}000x_1 + 1{,}000x_2 + 1{,}000x_3 + 1{,}000x_4 + 1{,}000x_5 \\ & + 2{,}000y_1 + 2{,}000y_2 + 2{,}000y_3 + 2{,}000y_4 + 2{,}000y_5\end{aligned}$$

What constraints does CSL face? Note that we are given $y_1 = 50$, and that for $t = 1, 2, 3, 4, 5$, CSL must ensure that

$$\begin{aligned}&\text{Number of available technician hours during month } t \\ &\qquad \geq \text{Number of technician hours required during month } t\end{aligned} \qquad (72)$$

Because each trainee requires 50 hours of experienced technician time, and each skilled technician is available for 160 hours per month,

$$\text{Number of available technician hours during month } t = 160y_t - 50x_t$$

Now (72) yields the following five constraints:

$$\begin{aligned}
160y_1 - 50x_1 &\geq 6{,}000 && \text{(month 1 constraint)} \\
160y_2 - 50x_2 &\geq 7{,}000 && \text{(month 2 constraint)} \\
160y_3 - 50x_3 &\geq 8{,}000 && \text{(month 3 constraint)} \\
160y_4 - 50x_4 &\geq 9{,}500 && \text{(month 4 constraint)} \\
160y_5 - 50x_5 &\geq 11{,}000 && \text{(month 5 constraint)}
\end{aligned}$$

As in the other multiperiod formulations, we need constraints that relate variables from different periods. In the CSL problem, it is important to realize that the number of skilled technicians available at the beginning of any month is determined by the number of skilled technicians available during the previous month and the number of technicians trained during the previous month:

$$\begin{array}{l} \text{Experienced technicians available} \\ \text{at beginning of month } t \end{array} = \begin{array}{l} \text{Experienced technicians available} \\ \text{at beginning of month } (t-1) \\ + \text{ technicians trained during month } (t-1) \\ - \text{ experienced technicians who quit during} \\ \text{month } (t-1) \end{array} \qquad (73)$$

For example, for February, (73) yields

$$y_2 = y_1 + x_1 - 0.05y_1 \qquad \text{or} \qquad y_2 = 0.95y_1 + x_1$$

Similarly, for March, (73) yields

$$y_3 = 0.95y_2 + x_2$$

and for April,

$$y_4 = 0.95y_3 + x_3$$

and for May,

$$y_5 = 0.95y_4 + x_4$$

Adding the sign restrictions $x_t \geq 0$ and $y_t \geq 0$ ($t = 1, 2, 3, 4, 5$), we obtain the following LP:

$$\begin{array}{lll} \min z = & 1{,}000x_1 + 1{,}000x_2 + 1{,}000x_3 + 1{,}000x_4 + 1{,}000x_5 & \\ & + 2{,}000y_1 + 2{,}000y_2 + 2{,}000y_3 + 2{,}000y_4 + 2{,}000y_5 & \\ \text{s.t.} & 160y_1 - 50x_1 \geq 6{,}000 & y_1 = 50 \\ & 160y_2 - 50x_2 \geq 7{,}000 & 0.95y_1 + x_1 = y_2 \\ & 160y_3 - 50x_3 \geq 8{,}000 & 0.95y_2 + x_2 = y_3 \\ & 160y_4 - 50x_4 \geq 9{,}500 & 0.95y_3 + x_3 = y_4 \\ & 160y_5 - 50x_5 \geq 11{,}000 & 0.95y_4 + x_4 = y_5 \\ & x_t, y_t \geq 0 \qquad (t = 1, 2, 3, 4, 5) & \end{array}$$

The optimal solution is $z = 593{,}777$; $x_1 = 0$; $x_2 = 8.45$; $x_3 = 11.45$; $x_4 = 9.52$; $x_5 = 0$; $y_1 = 50$; $y_2 = 47.5$; $y_3 = 53.58$; $y_4 = 62.34$; and $y_5 = 68.75$.

In reality, the y_t's must be integers, so our solution is difficult to interpret. The problem with our formulation is that assuming that exactly 5% of the employees quit each month can cause the number of employees to change from an integer during one month to a fraction during the next month. We might want to assume that the number of employees quitting each month is the integer closest to 5% of the total workforce, but then we do not have a linear programming problem!

PROBLEMS

Group A

1 If $y_1 = 38$, then what would be the optimal solution to CSL's problem?

2 An insurance company believes that it will require the following numbers of personal computers during the next six months: January, 9; February, 5; March, 7; April, 9; May, 10; June, 5. Computers can be rented for a period of one, two, or three months at the following unit rates: one-month rate, \$200; two-month rate, \$350; three-month rate, \$450. Formulate an LP that can be used to minimize the cost of renting the required computers. You may assume that if a machine is rented for a period of time extending beyond June, the cost of the rental should be prorated. For example, if a computer is rented for three months at the beginning of May, then a rental fee of $\frac{2}{3}(450) = \$300$, not \$450, should be assessed in the objective function.

3 The IRS has determined that during each of the next 12 months it will need the number of supercomputers given in Table 45. To meet these requirements, the IRS rents

TABLE 45

Month	Computer Requirements
1	800
2	1,000
3	600
4	500
5	1,200
6	400
7	800
8	600
9	400
10	500
11	800
12	600

supercomputers for a period of one, two, or three months. It costs \$100 to rent a supercomputer for one month, \$180 for two months, and \$250 for three months. At the beginning of month 1, the IRS has no supercomputers. Determine the rental plan that meets the next 12 months' requirements at minimum cost. *Note:* You may assume that fractional rentals are okay, so if your solution says to rent 140.6 computers for one month we can round this up or down (to 141 or 140) without having much effect on the total cost.

Group B

4 You own a wheat warehouse with a capacity of 20,000 bushels. At the beginning of month 1, you have 6,000 bushels of wheat. Each month, wheat can be bought and sold at the price per 1000 bushels given in Table 46.

TABLE 46

Month	Selling Price (\$)	Purchase Price (\$)
1	3	8
2	6	8
3	7	2
4	1	3
5	4	4
6	5	3
7	5	3
8	1	2
9	3	5
10	2	5

The sequence of events during each month is as follows:

a You observe your initial stock of wheat.

b You can sell any amount of wheat up to your initial stock at the current month's selling price.

c You can buy (at the current month's buying price) as much wheat as you want, subject to the warehouse size limitation.

Your goal is to formulate an LP that can be used to determine how to maximize the profit earned over the next 10 months.

SUMMARY

Linear Programming Definitions

A **linear programming problem (LP)** consists of three parts:

1 A linear function (the **objective function**) of decision variables (say, $x_1, x_2, \ldots, x_n$) that is to be maximized or minimized.

2 A set of **constraints** (each of which must be a linear equality or linear inequality) that restrict the values that may be assumed by the decision variables.

3 The **sign restrictions,** which specify for each decision variable x_j either (1) variable x_j must be nonnegative—$x_j \geq 0$; or (2) variable x_j may be positive, zero, or negative—x_j is **unrestricted in sign (urs).**

The coefficient of a variable in the objective function is the variable's **objective function coefficient.** The coefficient of a variable in a constraint is a **technological coefficient.** The right-hand side of each constraint is called a **right-hand side (rhs).**

A *point* is simply a specification of the values of each decision variable. The **feasible region** of an LP consists of all points satisfying the LP's constraints and sign restrictions. Any point in the feasible region that has the largest z-value of all points in the feasible region (for a max problem) is an **optimal solution** to the LP. An LP may have no optimal solution, one optimal solution, or an infinite number of optimal solutions.

A constraint in an LP is **binding** if the left-hand side and the right-hand side are equal when the values of the variables in the optimal solution are substituted into the constraint.

Graphical Solution of Linear Programming Problems

The feasible region for any LP is a **convex set.** If an LP has an optimal solution, there is an extreme (or corner) point of the feasible region that is an optimal solution to the LP.

We may graphically solve an LP (max problem) with two decision variables as follows:

Step 1 Graph the feasible region.

Step 2 Draw an isoprofit line.

Step 3 Move parallel to the isoprofit line in the direction of increasing *z*. The last point in the feasible region that contacts an isoprofit line is an optimal solution to the LP.

LP Solutions: Four Cases

When an LP is solved, one of the following four cases will occur:

Case 1 The LP has a unique solution.

Case 2 The LP has more than one (actually an infinite number of) optimal solutions. This is the case of **alternative optimal solutions.** Graphically, we recognize this case when the isoprofit line last hits an entire line segment before leaving the feasible region.

Case 3 The LP is **infeasible** (it has no feasible solution). This means that the feasible region contains no points.

Case 4 The LP is unbounded. This means (in a max problem) that there are points in the feasible region with arbitrarily large *z*-values. Graphically, we recognize this case by the fact that when we move parallel to an isoprofit line in the direction of increasing *z*, we never lose contact with the LP's feasible region.

Formulating LPs

The most important step in formulating most LPs is to determine the decision variables correctly.

In any constraint, the terms must have the same units. For example, one term cannot have the units "pounds of raw material" while another term has the units "ounces of raw material."

REVIEW PROBLEMS

Group A

1 Bloomington Breweries produces beer and ale. Beer sells for \$5 per barrel, and ale sells for \$2 per barrel. Producing a barrel of beer requires 5 lb of corn and 2 lb of hops. Producing a barrel of ale requires 2 lb of corn and 1 lb of hops. Sixty pounds of corn and 25 lb of hops are available. Formulate an LP that can be used to maximize revenue. Solve the LP graphically.

2 Farmer Jones bakes two types of cake (chocolate and vanilla) to supplement his income. Each chocolate cake can be sold for \$1, and each vanilla cake can be sold for 50¢. Each chocolate cake requires 20 minutes of baking time and uses 4 eggs. Each vanilla cake requires 40 minutes of baking time and uses 1 egg. Eight hours of baking time and 30 eggs are available. Formulate an LP to maximize Farmer Jones's revenue, then graphically solve the LP. (A fractional number of cakes is okay.)

3 I now have $100. The following investments are available during the next three years:

Investment A Every dollar invested now yields $0.10 a year from now and $1.30 three years from now.
Investment B Every dollar invested now yields $0.20 a year from now and $1.10 two years from now.
Investment C Every dollar invested a year from now yields $1.50 three years from now.

During each year, uninvested cash can be placed in money market funds, which yield 6% interest per year. At most $50 may be placed in each of investments A, B, and C. Formulate an LP to maximize my cash on hand three years from now.

4 Sunco processes oil into aviation fuel and heating oil. It costs $40 to purchase each 1,000 barrels of oil, which is then distilled and yields 500 barrels of aviation fuel and 500 barrels of heating oil. Output from the distillation may be sold directly or processed in the catalytic cracker. If sold after distillation without further processing, aviation fuel sells for $60 per 1,000 barrels, and heating oil sells for $40 per 1,000 barrels. It takes 1 hour to process 1,000 barrels of aviation fuel in the catalytic cracker, and these 1,000 barrels can be sold for $130. It takes 45 minutes to process 1,000 barrels of heating oil in the cracker, and these 1,000 barrels can be sold for $90. Each day, at most 20,000 barrels of oil can be purchased, and 8 hours of cracker time are available. Formulate an LP to maximize Sunco's profits.

5 Finco has the following investments available:

Investment A For each dollar invested at time 0, we receive $0.10 at time 1 and $1.30 at time 2. (Time 0 = now; time 1 = one year from now; and so on.)
Investment B For each dollar invested at time 1, we receive $1.60 at time 2.
Investment C For each dollar invested at time 2, we receive $1.20 at time 3.

At any time, leftover cash may be invested in T-bills, which pay 10% per year. At time 0, we have $100. At most, $50 can be invested in each of investments A, B, and C. Formulate an LP that can be used to maximize Finco's cash on hand at time 3.

6 All steel manufactured by Steelco must meet the following requirements: 3.2–3.5% carbon; 1.8–2.5% silicon; 0.9–1.2% nickel; tensile strength of at least 45,000 pounds per square inch (psi). Steelco manufactures steel by combining two alloys. The cost and properties of each alloy are given in Table 47. Assume that the tensile strength of a mixture of the two alloys can be determined by averaging that of the alloys that are mixed together. For example, a one-ton mixture that is 40% alloy 1 and 60% alloy 2 has a tensile strength of 0.4(42,000) + 0.6(50,000). Use linear programming to determine how to minimize the cost of producing a ton of steel.

TABLE 47

	Alloy 1	Alloy 2
Cost per ton ($)	$190	$200
Percent silicon	2	2.5
Percent nickel	1	1.5
Percent carbon	3	4
Tensile strength (psi)	42,000	50,000

7 Steelco manufactures two types of steel at three different steel mills. During a given month, each steel mill has 200 hours of blast furnace time available. Because of differences in the furnaces at each mill, the time and cost to produce a ton of steel differs for each mill. The time and cost for each mill are shown in Table 48. Each month, Steelco must manufacture at least 500 tons of steel 1 and 600 tons of steel 2. Formulate an LP to minimize the cost of manufacturing the desired steel.

8[†] Walnut Orchard has two farms that grow wheat and corn. Because of differing soil conditions, there are differences in the yields and costs of growing crops on the two farms. The yields and costs are shown in Table 49. Each farm has 100 acres available for cultivation; 11,000 bushels of wheat and 7,000 bushels of corn must be grown. Determine a planting plan that will minimize the cost of meeting these demands. How could an extension of this model be used to allocate crop production efficiently throughout a nation?

9 Candy Kane Cosmetics (CKC) produces Leslie Perfume, which requires chemicals and labor. Two production processes are available: Process 1 transforms 1 unit of labor and 2 units of chemicals into 3 oz of perfume. Process 2 transforms 2 units of labor and 3 units of chemicals into 5 oz of perfume. It costs CKC $3 to purchase a unit of labor and $2 to purchase a unit of chemicals. Each year, up to 20,000 units of labor and 35,000 units of chemicals can be purchased. In the absence of advertising, CKC believes it can sell 1,000 oz of perfume. To stimulate demand for

TABLE 48
Producing a Ton of Steel

	Steel 1		Steel 2	
Mill	Cost	Time (Minutes)	Cost	Time (Minutes)
1	$10	20	$11	22
2	$12	24	$ 9	18
3	$14	28	$10	30

TABLE 49

	Farm 1	Farm 2
Corn yield/acre (bushels)	500	650
Cost/acre of corn ($)	100	120
Wheat yield/acre (bushels)	400	350
Cost/acre of wheat ($)	90	80

[†]Based on Heady and Egbert (1964).

Leslie, CKC can hire the lovely model Jenny Nelson. Jenny is paid $100/hour. Each hour Jenny works for the company is estimated to increase the demand for Leslie Perfume by 200 oz. Each ounce of Leslie Perfume sells for $5. Use linear programming to determine how CKC can maximize profits.

10 Carco has a $150,000 advertising budget. To increase automobile sales, the firm is considering advertising in newspapers and on television. The more Carco uses a particular medium, the less effective is each additional ad. Table 50 shows the number of new customers reached by each ad. Each newspaper ad costs $1,000, and each television ad costs $10,000. At most, 30 newspaper ads and 15 television ads can be placed. How can Carco maximize the number of new customers created by advertising?

11 Sunco Oil has refineries in Los Angeles and Chicago. The Los Angeles refinery can refine up to 2 million barrels of oil per year, and the Chicago refinery up to 3 million. Once refined, oil is shipped to two distribution points: Houston and New York City. Sunco estimates that each distribution point can sell up to 5 million barrels per year. Because of differences in shipping and refining costs, the profit earned (in dollars) per million barrels of oil shipped depends on where the oil was refined and on the point of distribution (see Table 51). Sunco is considering expanding the capacity of each refinery. Each million barrels of annual refining capacity that is added will cost $120,000 for the Los Angeles refinery and $150,000 for the Chicago refinery. Use linear programming to determine how Sunco can maximize its profits less expansion costs over a ten-year period.

12 For a telephone survey, a marketing research group needs to contact at least 150 wives, 120 husbands, 100 single adult males, and 110 single adult females. It costs $2 to make a daytime call and (because of higher labor costs) $5 to make an evening call. Table 52 lists the results. Because of limited staff, at most half of all phone calls can be evening calls. Formulate an LP to minimize the cost of completing the survey.

TABLE 50

	Number of Ads	New Customers
Newspaper	1–10	900
	11–20	600
	21–30	300
Television	1–5	10,000
	6–10	5,000
	11–15	2,000

TABLE 51

	Profit per Million Barrels ($)	
From	To Houston	To New York
Los Angeles	20,000	15,000
Chicago	18,000	17,000

TABLE 52

Person Responding	Percent of Daytime Calls	Percent of Evening Calls
Wife	30	30
Husband	10	30
Single male	10	15
Single female	10	20
None	40	5

13 Feedco produces two types of cattle feed, both consisting totally of wheat and alfalfa. Feed 1 must contain at least 80% wheat, and feed 2 must contain at least 60% alfalfa. Feed 1 sells for $1.50/lb, and feed 2 sells for $1.30/lb. Feedco can purchase up to 1,000 lb of wheat at 50¢/lb and up to 800 lb of alfalfa at 40¢/lb. Demand for each type of feed is unlimited. Formulate an LP to maximize Feedco's profit.

14 Feedco (see Problem 13) has decided to give its customer (assume it has only one customer) a quantity discount. If the customer purchases more than 300 lb of feed 1, each pound over the first 300 lb will sell for only $1.25/lb. Similarly, if the customer purchases more than 300 pounds of feed 2, each pound over the first 300 lb will sell for $1.00/lb. Modify the LP of Problem 13 to account for the presence of quantity discounts. (*Hint:* Define variables for the feed sold at each price.)

15 Chemco produces two chemicals: A and B. These chemicals are produced via two manufacturing processes. Process 1 requires 2 hours of labor and 1 lb of raw material to produce 2 oz of A and 1 oz of B. Process 2 requires 3 hours of labor and 2 lb of raw material to produce 3 oz of A and 2 oz of B. Sixty hours of labor and 40 lb of raw material are available. Demand for A is unlimited, but only 20 oz of B can be sold. A sells for $16/oz, and B sells for $14/oz. Any B that is unsold must be disposed of at a cost of $2/oz. Formulate an LP to maximize Chemco's revenue less disposal costs.

16 Suppose that in the CSL computer example of Section 3.12, it takes two months to train a technician and that during the second month of training, each trainee requires 10 hours of experienced technician time. Modify the formulation in the text to account for these changes.

17 Furnco manufactures tables and chairs. Each table and chair must be made entirely out of oak or entirely out of pine. A total of 150 board ft of oak and 210 board ft of pine are available. A table requires either 17 board ft of oak or 30 board ft of pine, and a chair requires either 5 board ft of oak or 13 board ft of pine. Each table can be sold for $40, and each chair for $15. Formulate an LP that can be used to maximize revenue.

18† The city of Busville contains three school districts. The number of minority and nonminority students in each district is given in Table 53. Of all students, 25% ($\frac{200}{800}$) are minority students.

†Based on Franklin and Koenigsberg (1973).

TABLE 53

District	Minority Students	Nonminority Students
1	50	200
2	50	250
3	100	150

TABLE 54

District	Cooley High	Walt Whitman High
1	1	2
2	2	1
3	1	1

The local court has decided that both of the town's two high schools (Cooley High and Walt Whitman High) must have approximately the same percentage of minority students (within ±5%) as the entire town. The distances (in miles) between the school districts and the high schools are given in Table 54. Each high school must have an enrollment of 300–500 students. Use linear programming to determine an assignment of students to schools that minimizes the total distance students must travel to school.

19[†] Brady Corporation produces cabinets. Each week, it requires 90,000 cu ft of processed lumber. The company may obtain lumber in two ways. First, it may purchase lumber from an outside supplier and then dry it in the supplier's kiln. Second, it may chop down logs on its own land, cut them into lumber at its sawmill, and finally dry the lumber in its own kiln. Brady can purchase grade 1 or grade 2 lumber. Grade 1 lumber costs $3 per cu ft and when dried yields 0.7 cu ft of useful lumber. Grade 2 lumber costs $7 per cubic foot and when dried yields 0.9 cu ft of useful lumber. It costs the company $3 to chop down a log. After being cut and dried, a log yields 0.8 cu ft of lumber. Brady incurs costs of $4 per cu ft of lumber dried. It costs $2.50 per cu ft of logs sent through the sawmill. Each week, the sawmill can process up to 35,000 cu ft of lumber. Each week, up to 40,000 cu ft of grade 1 lumber and up to 60,000 cu ft of grade 2 lumber can be purchased. Each week, 40 hours of time are available for drying lumber. The time it takes to dry 1 cu ft of grade 1 lumber, grade 2 lumber, or logs is as follows: grade 1—2 seconds; grade 2—0.8 second; log—1.3 seconds. Formulate an LP to help Brady minimize the weekly cost of meeting the demand for processed lumber.

20[‡] The Canadian Parks Commission controls two tracts of land. Tract 1 consists of 300 acres and tract 2, 100 acres. Each acre of tract 1 can be used for spruce trees or hunting, or both. Each acre of tract 2 can be used for spruce trees or camping, or both. The capital (in hundreds of dollars) and labor (in worker-days) required to maintain one acre of each tract, and the profit (in thousands of dollars) per acre for each possible use of land are given in Table 55. Capital of $150,000 and 200 man-days of labor are available. How should the land be allocated to various uses to maximize profit received from the two tracts?

[†]Based on Carino and Lenoir (1988).
[‡]Based on Cheung and Auger (1976).

21[§] Chandler Enterprises produces two competing products: A and B. The company wants to sell these products to two groups of customers: group 1 and group 2. The value each customer places on a unit of A and B is as shown in Table 56. Each customer will buy either product A or product B, but not both. A customer is willing to buy product A if she believes that

$$\text{Value of product A} - \text{price of product A} \geq \text{Value of product B} - \text{price of product B}$$

and

$$\text{Value of product A} - \text{price of product A} \geq 0$$

A customer is willing to buy product B if she believes that

$$\text{Value of product B} - \text{price of product B} \geq \text{value of product A} - \text{price of product A}$$

and

$$\text{Value of product B} - \text{price of product B} \geq 0$$

Group 1 has 1,000 members, and group 2 has 1,500 members. Chandler wants to set prices for each product that ensure that group 1 members purchase product A and group 2 members purchase product B. Formulate an LP that will help Chandler maximize revenues.

22[¶] Alden Enterprises produces two products. Each product can be produced on one of two machines. The length of time needed to produce each product (in hours) on each machine is as shown in Table 57. Each month, 500 hours of time are available on each machine. Each month, customers are willing to buy up to the quantities of each product at the

TABLE 55

Tract	Capital	Labor	Profit
1 Spruce	3	0.1	0.2
1 Hunting	3	0.2	0.4
1 Both	4	0.2	0.5
2 Spruce	1	0.05	0.06
2 Camping	30	5	0.09
2 Both	10	1.01	1.1

TABLE 56

	Group 1 Customer	Group 2 Customer
Value of A to	$10	$12
Value of B to	$8	$15

[§]Based on Dobson and Kalish (1988).
[¶]Based on Jain, Stott, and Vasold (1978).

TABLE 57

Product	Machine 1	Machine 2
1	4	3
2	7	4

TABLE 58

	Demands		Prices	
Product	Month 1	Month 2	Month 1	Month 2
1	100	190	$55	$12
2	140	130	$65	$32

prices given in Table 58. The company's goal is to maximize the revenue obtained from selling units during the next two months. Formulate an LP to help meet this goal.

23 Kiriakis Electronics produces three products. Each product must be processed on each of three types of machines. When a machine is in use, it must be operated by a worker. The time (in hours) required to process each product on each machine and the profit associated with each product are shown in Table 59. At present, five type 1 machines, three type 2 machines, and four type 3 machines are available. The company has 10 workers available and must determine how many workers to assign to each machine. The plant is open 40 hours per week, and each worker works 35 hours per week. Formulate an LP that will enable Kiriakis to assign workers to machines in a way that maximizes weekly profits. (*Note:* A worker need not spend the entire work week operating a single machine.)

24 Gotham City Hospital serves cases from four diagnostic-related groups (DRGs). The profit contribution, diagnostic service use (in hours), bed-day use (in days), nursing care use (in hours), and drug use (in dollars) are given in Table 60. The hospital now has available each week 570 hours of diagnostic services, 1,000 bed-days, 50,000 nursing hours, and $50,000 worth of drugs. To meet the community's minimum health care demands at least 10 DRG1, 15 DRG2, 40 DRG3, and 160 DRG4 cases must be handled each week. Use LP to determine the hospital's optimal mix of DRGs.†

TABLE 59

	Product 1	Product 2	Product 3
Machine 1	2	3	4
Machine 2	3	5	6
Machine 3	4	7	9
Profit ($)	6	8	10

TABLE 60

DRG	Profit	Diagnostic Services	Bed-Day	Nursing Use	Drugs
1	2,000	7	5	30	800
2	1,500	4	2	10	500
3	500	2	1	5	150
4	300	1	0	1	50

25 Oliver Winery produces four award-winning wines in Bloomington, Indiana. The profit contribution, labor hours, and tank usage (in hours) per gallon for each type of wine are given in Table 61. By law, at most 100,000 gallons of wine can be produced each year. A maximum of 12,000 labor hours and 32,000 tank hours are available annually. Each gallon of wine 1 spends an average of $\frac{1}{3}$ year in inventory; wine 2, an average of 1 year; wine 3, an average of 2 years; wine 4, an average of 3.333 years. The winery's warehouse can handle an average inventory level of 50,000 gallons. Determine how much of each type of wine should be produced annually to maximize Oliver Winery's profit.

26 Graphically solve the following LP:

$$\begin{aligned}
\min z = 5x_1 &+ x_2 \\
\text{s.t.} \quad 2x_1 &+ x_2 \geq 6 \\
x_1 &+ x_2 \geq 4 \\
2x_1 &+ 10x_2 \geq 20 \\
x_1, &\ x_2 \geq 0
\end{aligned}$$

27 Grummins Engine produces diesel trucks. New government emission standards have dictated that the average pollution emissions of all trucks produced in the next three years cannot exceed 10 grams per truck. Grummins produces two types of trucks. Each type 1 truck sells for $20,000, costs $15,000 to manufacture, and emits 15 grams of pollution. Each type 2 truck sells for $17,000, costs $14,000 to manufacture, and emits 5 grams of pollution. Production capacity limits total truck production during each year to at most 320 trucks. Grummins knows that the maximum number of each truck type that can be sold during each of the next three years is given in Table 62.

Thus, *at most,* 300 type 1 trucks can be sold during year 3. Demand may be met from previous production or the current year's production. It costs $2,000 to hold 1 truck (of any type) in inventory for one year. Formulate an LP to help Grummins maximize its profit during the next three years.

TABLE 61

Wine	Profit ($)	Labor (Hr)	Tank (Hr)
1	6	.2	.5
2	12	.3	.5
3	20	.3	1
4	30	.5	1.5

†Based on Robbins and Tuntiwonpiboon (1989).

TABLE 62
Maximum Demand for Trucks

Year	Type 1	Type 2
1	100	200
2	200	100
3	300	150

TABLE 63

Shift	Hourly Salary	Defects (per Capacitor)	Price
8 A.M.–4 P.M.	\$12	4	\$18
4 P.M.–Midnight	\$16	3	\$22
Midnight–8 A.M.	\$20	2	\$24

28 Describe all optimal solutions to the following LP:

$$\begin{aligned} \min z &= 4x_1 + x_2 \\ \text{s.t.} \quad 3x_1 + x_2 &\geq 6 \\ 4x_1 + x_2 &\geq 12 \\ x_1 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

29 Juiceco manufactures two products: premium orange juice and regular orange juice. Both products are made by combining two types of oranges: grade 6 and grade 3. The oranges in premium juice must have an average grade of at least 5, those in regular juice, at least 4. During each of the next two months Juiceco can sell up to 1,000 gallons of premium juice and up to 2,000 gallons of regular juice. Premium juice sells for \$1.00 per gallon, while regular juice sells for 80¢ per gallon. At the beginning of month 1, Juiceco has 3,000 gallons of grade 6 oranges and 2,000 gallons of grade 3 oranges. At the beginning of month 2, Juiceco may purchase additional grade 3 oranges for 40¢ per gallon and additional grade 6 oranges for 60¢ per gallon. Juice spoils at the end of the month, so it makes no sense to make extra juice during month 1 in the hopes of using it to meet month 2 demand. Oranges left at the end of month 1 may be used to produce juice for month 2. At the end of month 1 a holding cost of 5¢ is assessed against each gallon of leftover grade 3 oranges, and 10¢ against each gallon of leftover grade 6 oranges. In addition to the cost of the oranges, it costs 10¢ to produce each gallon of (regular or premium) juice. Formulate an LP that could be used to maximize the profit (revenues − costs) earned by Juiceco during the next two months.

30 Graphically solve the following linear programming problem:

$$\begin{aligned} \max z &= 5x_1 - x_2 \\ \text{s.t.} \quad 2x_1 + 3x_2 &\geq 12 \\ x_1 - 3x_2 &\geq 0 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

31 Graphically find all solutions to the following LP:

$$\begin{aligned} \min z &= x_1 - 2x_2 \\ \text{s.t.} \quad x_1 &\geq 4 \\ x_1 + x_2 &\geq 8 \\ x_1 - x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

32 Each day Eastinghouse produces capacitors during three shifts: 8 A.M.–4 P.M., 4 P.M.–midnight, midnight–8 A.M. The hourly salary paid to the employees on each shift, the price charged for each capacitor made during each shift, and the number of defects in each capacitor produced during a given shift are shown in Table 63. Each of the company's 25 workers can be assigned to one of the three shifts. A worker produces 10 capacitors during a shift, but because of machinery limitations, no more than 10 workers can be assigned to any shift. Each day, at most 250 capacitors can be sold, and the average number of defects per capacitor for the day's production cannot exceed three. Formulate an LP to maximize Eastinghouse's daily profit (sales revenue − labor cost).

33 Graphically find all solutions to the following LP:

$$\begin{aligned} \max z &= 4x_1 + x_2 \\ \text{s.t.} \quad 8x_1 + 2x_2 &\leq 16 \\ x_1 + x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

34 During the next three months Airco must meet (on time) the following demands for air conditioners: month 1, 300; month 2, 400; month 3, 500. Air conditioners can be produced in either New York or Los Angeles. It takes 1.5 hours of skilled labor to produce an air conditioner in Los Angeles, and 2 hours in New York. It costs \$400 to produce an air conditioner in Los Angeles, and \$350 in New York. During each month, each city has 420 hours of skilled labor available. It costs \$100 to hold an air conditioner in inventory for a month. At the beginning of month 1, Airco has 200 air conditioners in stock. Formulate an LP whose solution will tell Airco how to minimize the cost of meeting air conditioner demands for the next three months.

35 Formulate the following as a linear programming problem: A greenhouse operator plans to bid for the job of providing flowers for city parks. He will use tulips, daffodils, and flowering shrubs in three types of layouts. A Type 1 layout uses 30 tulips, 20 daffodils, and 4 flowering shrubs. A Type 2 layout uses 10 tulips, 40 daffodils, and 3 flowering shrubs. A Type 3 layout uses 20 tulips, 50 daffodils, and 2 flowering shrubs. The net profit is \$50 for each Type 1 layout, \$30 for each Type 2 layout, and \$60 for each Type 3 layout. He has 1,000 tulips, 800 daffodils, and 100 flowering shrubs. How many layouts of each type should be used to yield maximum profit?

36 Explain how your formulation in Problem 35 changes if both of the following conditions are added:

a The number of Type 1 layouts cannot exceed the number of Type 2 layouts.

b There must be at least five layouts of each type.

37 Graphically solve the following LP problem:

$$\begin{aligned} \min z &= 6x_1 + 2x_2 \\ \text{s.t.} \quad 3x_1 + 2x_2 &\geq 12 \\ 2x_1 + 4x_2 &\geq 12 \\ x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

38 We produce two products: product 1 and product 2 on two machines (machine 1 and machine 2). The number of hours of machine time and labor depends on the machine and the product as shown in Table 64.

The cost of producing a unit of each product is shown in Table 65.

The number of labor hours and machine time available this month are in Table 66.

This month, at least 200 units of product 1 and at least 240 units of product 2 must be produced. Also, at least half of product 1 must be made on machine 1, and at least half of product 2 must be made on machine 2. Determine how we can minimize the cost of meeting our monthly demands.

39 Carrotco manufactures two products: 1 and 2. Each unit of each product must be processed on machine 1 and machine 2 and uses raw material 1 and raw material 2. The resource usage is as in Table 67.

TABLE 64

	Product 1 Machine 1	Product 2 Machine 1	Product 1 Machine 2	Product 2 Machine 2
Machine time	0.7	0.75	0.8	0.9
Labor	0.75	0.75	1.2	1

TABLE 65

Product 1 Machine 1	Product 2 Machine 1	Product 1 Machine 2	Product 2 Machine 2
\$1.50	\$0.40	\$2.20	\$4.00

TABLE 66

Resource	Hours Available
Machine 1	200
Machine 2	200
Labor	400

TABLE 67

	Product 1	Product 2
Machine 1	0.6	0.4
Machine 2	0.4	0.3
Raw material 1	2	1
Raw material 2	1	2

TABLE 68

	Product 1	Product 2
Demand	400	300
Sales Price	30	35

Thus, producing one unit of product 1 uses .6 unit of machine 1 time, .4 unit of machine 2 time, 2 units of raw material 1, and 1 unit of raw material 2. The sales price per unit and demand for each product are in Table 68.

It costs \$4 to purchase each unit of raw material 1 and \$5 to produce each unit of raw material 2. Unlimited amounts of raw material can be purchased. Two hundred units of machine 1 time and 300 units of machine 2 time are available. Determine how Carrotco can maximize its profit.

40 A company assembles two products: A and B. Product A sells for \$11 per unit, and product B sells for \$23 per unit. A unit of product A requires 2 hours on assembly line 1 and 1 unit of raw material. A unit of product B requires 2 units of raw material, 1 unit of A, and 2 hours on line 2. For line 1, 1,300 hours of time are available and 500 hours of time are available on line 2. A unit of raw material may be bought (for \$5 a unit) or produced (at no cost) by using 2 hours of time on line 1. Determine how to maximize profit.

41 Ann and Ben are getting divorced and want to determine how to divide their joint property: retirement account, home, summer cottage, investments, and miscellaneous assets. To begin, Ann and Ben are told to allocate 100 total points to the assets. Their allocation is as shown in Table 69.

Assuming that all assets are divisible (that is, a fraction of each asset may be given to each person), how should the assets be allocated? Two criteria should govern the asset allocation:

Criteria 1 Each person should end up with the same number of points. This prevents Ann from envying Ben and Ben from envying Ann.

Criteria 2 The total number of points received by Ann and Ben should be maximized.

If assets could not be split between people, what problem arises?

42 Eli Daisy manufactures two drugs in Los Angeles and Indianapolis. The cost of manufacturing a pound of each drug is shown in Table 70.

TABLE 69

	Points	
Item	**Ann's**	**Ben's**
Retirement account	50	40
Home	20	30
Summer cottage	15	10
Investments	10	10
Miscellaneous	5	10

TABLE 70

City	Drug 1 Cost ($)	Drug 2 Cost ($)
Indianapolis	4.10	4.50
Los Angeles	4.00	5.20

TABLE 71

City	Drug 1 Time (Hr)	Drug 2 Time (Hr)
Indianapolis	.2	.3
Los Angeles	.24	.33

The machine time (in hours) required to produce a pound of each drug at each city is as in Table 71.

Daisy needs to produce at least 1,000 pounds of drug 1 and 2,000 pounds of drug 2 per week. The company has 500 hours per week of machine time in Indianapolis and 400 hours per week of machine time in Los Angeles. Determine how Lilly can minimize the cost of producing the needed drugs.

43 Daisy also produces Wozac in New York and Chicago. Each month, it can produce up to 30 units in New York and up to 35 units in Chicago. The cost of producing a unit each month at each location is shown in Table 72.

The customer demands shown in Table 73 must be met on time.

The cost of holding a unit in inventory (measured against ending inventory) is shown in Table 74.

TABLE 72

	Cost ($)	
Month	New York	Chicago
1	8.62	8.40
2	8.70	8.75
3	8.90	9.00

TABLE 73

Month	Demand (Units)
1	50
2	60
3	40

TABLE 74

Month	Holding Cost ($)
1	0.26
2	0.12
3	0.12

At the beginning of month 1, we have 10 units of Wozac in inventory. Determine a cost-minimizing schedule for the next three months.

44 You have been put in charge of the Dawson Creek oil refinery. The refinery produces gas and heating oil from crude oil. Gas sells for $11 per barrel and must have an average grade level of at least 9. Heating oil sells for $6 a barrel and must have an average grade level of at least 7. At most, 2,000 barrels of gas and 600 barrels of heating oil can be sold.

Incoming crude can be processed by one of three methods. The per barrel yield and per barrel cost of each processing method are shown in Table 75.

For example, if we refine one barrel of incoming crude by method 1, it costs us $3.40 and yields .2 barrels of grade 6, .2 barrels of grade 8, and .6 barrels of grade 10. These costs include the costs of buying the crude oil.

Before being processed into gas and heating oil, grades 6 and 8 may be sent through the catalytic cracker to improve their quality. For $1 per barrel, one barrel of grade 6 can be "cracked" into a barrel of grade 8. For $1.50 per barrel, a barrel of grade 8 can be cracked into a barrel of grade 10. Determine how to maximize the refinery's profit.

45 Currently we own 100 shares each of stocks 1 through 10. The original price we paid for these stocks, today's price, and the expected price in one year for each stock is shown in Table 76.

We need money today and are going to sell some of our stocks. The tax rate on capital gains is 30%. If we sell 50 shares of stock 1, then we must pay tax of $.3 \cdot 50(30 - 20) = \$150$. We must also pay transaction costs of 1% on each transaction. Thus, our sale of 50 shares of stock 1 would incur transaction costs of $.01 \cdot 50 \cdot 30 = \$15$. After taxes and transaction costs, we must be left with $30,000 from our stock sales. Our goal is to maximize the expected (before-tax) value in one year of our remaining stock. What stocks should we sell? Assume it is all right to sell a fractional share of stock.

Group B

46 Gotham City National Bank is open Monday–Friday from 9 A.M. to 5 P.M. From past experience, the bank knows that it needs the number of tellers shown in Table 77. The bank hires two types of tellers. Full-time tellers work 9–5 five days a week, except for 1 hour off for lunch. (The bank determines when a full-time employee takes lunch hour, but each teller must go between noon and 1 P.M. or between 1 P.M. and 2 P.M.) Full-time employees are paid (including fringe benefits) $8/hour (this includes payment for lunch hour). The bank may also hire part-time tellers. Each part-

TABLE 75

	Grade			
Method	6	8	10	Cost ($ per Barrel)
1	.2	.3	.5	3.40
2	.3	.4	.3	3.00
3	.4	.4	.2	2.60

TABLE 76

Stock	Shares Owned	Price ($) Purchase	Current	In One Year
1	100	20	30	36
2	100	25	34	39
3	100	30	43	42
4	100	35	47	45
5	100	40	49	51
6	100	45	53	55
7	100	50	60	63
8	100	55	62	64
9	100	60	64	66
10	100	65	66	70
Tax rate (%)	0.3			
Transaction cost (%)	0.01			

TABLE 77

Time Period	Tellers Required
9–10	4
10–11	3
11–Noon	4
Noon–1	6
1–2	5
2–3	6
3–4	8
4–5	8

time teller must work exactly 3 consecutive hours each day. A part-time teller is paid $5/hour (and receives no fringe benefits). To maintain adequate quality of service, the bank has decided that at most five part-time tellers can be hired. Formulate an LP to meet the teller requirements at minimum cost. Solve the LP on a computer. Experiment with the LP answer to determine an employment policy that comes close to minimizing labor cost.

47[†] The Gotham City Police Department employs 30 police officers. Each officer works 5 days per week. The crime rate fluctuates with the day of the week, so the number of police officers required each day depends on which day of the week it is: Saturday, 28; Sunday, 18; Monday, 18; Tuesday, 24; Wednesday, 25; Thursday, 16; Friday, 21. The police department wants to schedule police officers to minimize the number whose days off are not consecutive. Formulate an LP that will accomplish this goal. (*Hint:* Have a constraint for each day of the week that ensures that the proper number of officers are *not* working on the given day.)

[†]Based on Rothstein (1973).

48[‡]Alexis Cornby makes her living buying and selling corn. On January 1, she has 50 tons of corn and $1,000. On the first day of each month Alexis can buy corn at the following prices per ton: January, $300; February, $350; March, $400; April, $500. On the last day of each month, Alexis can sell corn at the following prices per ton: January, $250; February, $400; March, $350; April, $550. Alexis stores her corn in a warehouse that can hold at most 100 tons of corn. She must be able to pay cash for all corn at the time of purchase. Use linear programming to determine how Alexis can maximize her cash on hand at the end of April.

49[§]At the beginning of month 1, Finco has $400 in cash. At the beginning of months 1, 2, 3, and 4, Finco receives certain revenues, after which it pays bills (see Table 78). Any money left over may be invested for one month at the interest rate of 0.1% per month; for two months at 0.5% per month; for three months at 1% per month; or for four months at 2% per month. Use linear programming to determine an investment strategy that maximizes cash on hand at the beginning of month 5.

50 City 1 produces 500 tons of waste per day, and city 2 produces 400 tons of waste per day. Waste must be incinerated at incinerator 1 or 2, and each incinerator can process up to 500 tons of waste per day. The cost to incinerate waste is $40/ton at incinerator 1 and $30/ton at 2.

TABLE 78

Month	Revenues ($)	Bills ($)
1	400	600
2	800	500
3	300	500
4	300	250

[‡]Based on Charnes and Cooper (1955).
[§]Based on Robichek, Teichroew, and Jones (1965).

Incineration reduces each ton of waste to 0.2 tons of debris, which must be dumped at one of two landfills. Each landfill can receive at most 200 tons of debris per day. It costs $3 per mile to transport a ton of material (either debris or waste). Distances (in miles) between locations are shown in Table 79. Formulate an LP that can be used to minimize the total cost of disposing of the waste of both cities.

51[†] Silicon Valley Corporation (Silvco) manufactures transistors. An important aspect of the manufacture of transistors is the melting of the element germanium (a major component of a transistor) in a furnace. Unfortunately, the melting process yields germanium of highly variable quality.

Two methods can be used to melt germanium; method 1 costs $50 per transistor, and method 2 costs $70 per transistor. The qualities of germanium obtained by methods 1 and 2 are shown in Table 80. Silvco can refine melted germanium in an attempt to improve its quality. It costs $25 to refine the melted germanium for one transistor. The results of the refining process are shown in Table 81. Silvco has sufficient furnace capacity to melt or refine germanium for at most 20,000 transistors per month. Silvco's monthly demands are for 1,000 grade 4 transistors, 2,000 grade 3 transistors, 3,000 grade 2 transistors, and 3,000 grade 1 transistors. Use linear programming to minimize the cost of producing the needed transistors.

TABLE 79

City	Incinerator 1	Incinerator 2
1	30	5
2	36	42

Incinerator	Landfill 1	Landfill 2
1	5	8
2	9	6

TABLE 80

Grade of[‡] Melted Germanium	Percent Yielded by Melting	
	Method 1	Method 2
Defective	30	20
1	30	20
2	20	25
3	15	20
4	5	15

[‡]*Note:* Grade 1 is poor; grade 4 is excellent. The quality of the germanium dictates the quality of the manufactured transistor.

[†]Based on Smith (1965).

TABLE 81

Refined Grade of Germanium	Percent Yielded by Refining			
	Defective	Grade 1	Grade 2	Grade 3
Defective	30	0	0	0
1	25	30	0	0
2	15	30	40	0
3	20	20	30	50
4	10	20	30	50

TABLE 82

Input	Cost ($)	Pulp Content (%)
Box board	5	15
Tissue paper	6	20
Newsprint	8	30
Book paper	10	40

52[‡] A paper-recycling plant processes box board, tissue paper, newsprint, and book paper into pulp that can be used to produce three grades of recycled paper (grades 1, 2, and 3). The prices per ton and the pulp contents of the four inputs are shown in Table 82. Two methods, de-inking and asphalt dispersion, can be used to process the four inputs into pulp. It costs $20 to de-ink a ton of any input. The process of de-inking removes 10% of the input's pulp, leaving 90% of the original pulp. It costs $15 to apply asphalt dispersion to a ton of material. The asphalt dispersion process removes 20% of the input's pulp. At most, 3,000 tons of input can be run through the asphalt dispersion process or the de-inking process. Grade 1 paper can only be produced with newsprint or book paper pulp; grade 2 paper, only with book paper, tissue paper, or box board pulp; and grade 3 paper, only with newsprint, tissue paper, or box board pulp. To meet its current demands, the company needs 500 tons of pulp for grade 1 paper, 500 tons of pulp for grade 2 paper, and 600 tons of pulp for grade 3 paper. Formulate an LP to minimize the cost of meeting the demands for pulp.

53 Turkeyco produces two types of turkey cutlets for sale to fast-food restaurants. Each type of cutlet consists of white meat and dark meat. Cutlet 1 sells for $4/lb and must consist of at least 70% white meat. Cutlet 2 sells for $3/lb and must consist of at least 60% white meat. At most, 50 lb of cutlet 1 and 30 lb of cutlet 2 can be sold. The two types of turkey used to manufacture the cutlets are purchased from the GobbleGobble Turkey Farm. Each type 1 turkey costs $10 and yields 5 lb of white meat and 2 lb of dark meat. Each type 2 turkey costs $8 and yields 3 lb of white meat and 3 lb of dark meat. Formulate an LP to maximize Turkeyco's profit.

54 Priceler manufactures sedans and wagons. The number of vehicles that can be sold each of the next three months

[‡]Based on Glassey and Gupta (1975).

TABLE 83

Month	Sedans	Wagons
1	1,100	600
2	1,500	700
3	1,200	50

are listed in Table 83. Each sedan sells for \$8,000, and each wagon sells for \$9,000. It costs \$6,000 to produce a sedan and \$7,500 to produce a wagon. To hold a vehicle in inventory for one month costs \$150 per sedan and \$200 per wagon. During each month, at most 1,500 vehicles can be produced. Production line restrictions dictate that during month 1 at least two-thirds of all cars produced must be sedans. At the beginning of month 1, 200 sedans and 100 wagons are available. Formulate an LP that can be used to maximize Priceler's profit during the next three months.

55 The production-line employees at Grummins Engine work four days a week, 10 hours a day. Each day of the week, (at least) the following numbers of line employees are needed: Monday–Friday, 7 employees; Saturday and Sunday, 3 employees. Grummins has 11 production-line employees. Formulate an LP that can be used to maximize the number of consecutive days off received by the employees. For example, a worker who gets Sunday, Monday, and Wednesday off receives two consecutive days off.

56 Bank 24 is open 24 hours per day. Tellers work two consecutive 6-hour shifts and are paid \$10 per hour. The possible shifts are as follows: midnight–6 A.M., 6 A.M.–noon, noon–6 P.M., 6 P.M.–midnight. During each shift, the following numbers of customers enter the bank: midnight–6 A.M., 100; 6 A.M.–noon, 200; noon–6 P.M., 300; 6 P.M.–midnight, 200. Each teller can serve up to 50 customers per shift. To model a cost for customer impatience, we assume that any customer who is present at the end of a shift "costs" the bank \$5. We assume that by midnight of each day, all customers must be served, so each day's midnight–6 A.M. shift begins with 0 customers in the bank. Formulate an LP that can be used to minimize the sum of the bank's labor and customer impatience costs.

57[†] Transeast Airlines flies planes on the following route: L.A.–Houston–N.Y.–Miami–L.A. The length (in miles) of each segment of this trip is as follows: L.A.–Houston, 1,500 miles; Houston–N.Y., 1,700 miles; N.Y.–Miami, 1,300 miles; Miami–L.A., 2,700 miles. At each stop, the plane may purchase up to 10,000 gallons of fuel. The price of fuel at each city is as follows: L.A., 88¢; Houston, 15¢; N.Y., \$1.05; Miami, 95¢. The plane's fuel tank can hold at most 12,000 gallons. To allow for the possibility of circling over a landing site, we require that the ending fuel level for each leg of the flight be at least 600 gallons. The number of gallons used per mile on each leg of the flight is

$$1 + (\text{average fuel level on leg of flight}/2{,}000)$$

To simplify matters, assume that the average fuel level on any leg of the flight is

$$\frac{(\text{Fuel level at start of leg}) + (\text{fuel level at end of leg})}{2}$$

Formulate an LP that can be used to minimize the fuel cost incurred in completing the schedule.

[†]Based on Darnell and Loflin (1977).

58[‡] To process income tax forms, the IRS first sends each form through the data preparation (DP) department, where information is coded for computer entry. Then the form is sent to data entry (DE), where it is entered into the computer. During the next three weeks, the following number of forms will arrive: week 1, 40,000; week 2, 30,000; week 3, 60,000. The IRS meets the crunch by hiring employees who work 40 hours per week and are paid \$200 per week. Data preparation of a form requires 15 minutes, and data entry of a form requires 10 minutes. Each week, an employee is assigned to either data entry or data preparation. The IRS must complete processing of all forms by the end of week 5 and wants to minimize the cost of accomplishing this goal. Formulate an LP that will determine how many workers should be working each week and how the workers should be assigned over the next five weeks.

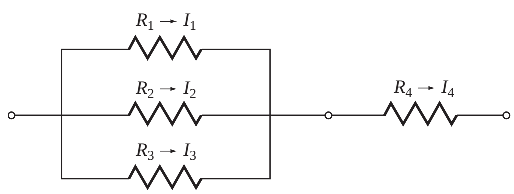
59 In the electrical circuit of Figure 11, I_t = current (in amperes) flowing through resistor t, V_t = voltage drop (in volts) across resistor t, and R_t = resistance (in ohms) of resistor t. Kirchoff's Voltage and Current Laws imply that $V_1 = V_2 = V_3$ and $I_1 + I_2 + I_3 = I_4$. The power dissipated by the current flowing through resistor t is $I_t^2R_t$. Ohm's Law implies that $V_t = I_tR_t$. The two parts of this problem should be solved independently.

a Suppose you are told that $I_1 = 4$, $I_2 = 6$, $I_3 = 8$, and $I_4 = 18$ are required. Also, the voltage drop across each resistor must be between 2 and 10 volts. Choose the R_t's to minimize the total dissipated power. Formulate an LP whose solution will solve your problem.

b Suppose you are told that $V_1 = 6$, $V_2 = 6$, $V_3 = 6$, and $V_4 = 4$ are required. Also, the current flowing through each resistor must be between 2 and 6 amperes. Choose the R_t's to minimize the total dissipated power. Formulate an LP whose solution will solve your problem. (*Hint:* Let $\frac{1}{R_t}$ ($t = 1, 2, 3, 4$) be your decision variables.)

60 The mayor of Llanview is trying to determine the number of judges needed to handle the judicial caseload.

FIGURE 11



[‡]Based on Lanzenauer et al. (1987).

TABLE 84

Month	Hours
January	400
February	300
March	200
April	600
May	800
June	300
July	200
August	400
September	300
October	200
November	100
December	300

During each month of the year it is estimated that the number of judicial hours needed is as given in Table 84.

a Each judge works all 12 months and can handle as many as 120 hours per month of casework. To avoid creating a backlog, all cases must be handled by the end of December. Formulate an LP whose solution will determine how many judges Llanview needs.

b If each judge received one month of vacation each year, how would your answer change?

Group C

61[†] E.J. Korvair Department Store has $1,000 in available cash. At the beginning of each of the next six months, E.J. will receive revenues and pay bills as shown in Table 85. It is clear that E.J. will have a short-term cash flow problem until the store receives revenues from the Christmas shopping season. To solve this problem, E.J. must borrow money.

At the beginning of July, E.J. may take out a six-month loan. Any money borrowed for a six-month period must be paid back at the end of December along with 9% interest (early payback does not reduce the interest cost of the loan). E.J. may also meet cash needs through month-to-month borrowing. Any money borrowed for a one-month period incurs an interest cost of 4% per month. Use linear programming to determine how E.J. can minimize the cost of paying its bills on time.

TABLE 85

Month	Revenues ($)	Bills ($)
July	1,000	5,000
August	2,000	5,000
September	2,000	6,000
October	4,000	2,000
November	7,000	2,000
December	9,000	1,000

[†]Based on Robichek, Teichroew, and Jones (1965).

62[‡] Olé Oil produces three products: heating oil, gasoline, and jet fuel. The average octane levels must be at least 4.5 for heating oil, 8.5 for gas, and 7.0 for jet fuel. To produce these products Olé purchases two types of oil: crude 1 (at $12 per barrel) and crude 2 (at $10 per barrel). Each day, at most 10,000 barrels of each type of oil can be purchased.

Before crude can be used to produce products for sale, it must be distilled. Each day, at most 15,000 barrels of oil can be distilled. It costs 10¢ to distill a barrel of oil. The result of distillation is as follows: (1) Each barrel of crude 1 yields 0.6 barrel of naphtha, 0.3 barrel of distilled 1, and 0.1 barrel of distilled 2. (2) Each barrel of crude 2 yields 0.4 barrel of naphtha, 0.2 barrel of distilled 1, and 0.4 barrel of distilled 2. Distilled naphtha can be used only to produce gasoline or jet fuel. Distilled oil can be used to produce heating oil or it can be sent through the catalytic cracker (at a cost of 15¢ per barrel). Each day, at most 5,000 barrels of distilled oil can be sent through the cracker. Each barrel of distilled 1 sent through the cracker yields 0.8 barrel of cracked 1 and 0.2 barrel of cracked 2. Each barrel of distilled 2 sent through the cracker yields 0.7 barrel of cracked 1 and 0.3 barrel of cracked 2. Cracked oil can be used to produce gasoline and jet fuel but not to produce heating oil.

The octane level of each type of oil is as follows: naphtha, 8; distilled 1, 4; distilled 2, 5; cracked 1, 9; cracked 2, 6.

All heating oil produced can be sold at $14 per barrel; all gasoline produced, $18 per barrel; and all jet fuel produced, $16 per barrel. Marketing considerations dictate that at least 3,000 barrels of each product must be produced daily. Formulate an LP to maximize Olé's daily profit.

63 Donald Rump is the international funds manager for Countribank. Each day Donald's job is to determine how the bank's current holdings of dollars, pounds, marks, and yen should be adjusted to meet the day's currency needs. Today the exchange rates between the various currencies are given in Table 86. For example, one dollar can be converted to .58928 pounds, or one pound can be converted to 1.697 dollars.

At the beginning of the day, Countribank has the currency holdings given in Table 87.

At the end of the day, Countribank must have at least the amounts of each currency given in Table 88.

Donald's goal is to each day transfer funds in a way that makes currency holdings satisfy the previously listed mini-

TABLE 86

	To			
From	Dollars	Pounds	Marks	Yen
Dollars	1	.58928	1.743	138.3
Pounds	1.697	1	2.9579	234.7
Marks	.57372	.33808	1	79.346
Yen	.007233	.00426	.0126	1

[‡]Based on Garvin et al. (1957).

TABLE 87

Currency	Units (in Billions)
Dollars	8
Pounds	1
Marks	8
Yen	0

TABLE 88

Currency	Units (in Billions)
Dollars	6
Pounds	3
Marks	1
Yen	10

mums, and maximizes the dollar value of the currency holdings at the end of the day.

To figure out the dollar value of, say, one pound, average the two conversion rates. Thus, one pound is worth approximately

$$\frac{1.697 + (1/.58928)}{2} = 1.696993 \text{ dollars}$$

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