(1) Consider the following linear system.

\[
\begin{align*}
2x_1 + x_2 + x_3 - 3x_4 + 9x_5 &= 7 \\
x_1 + x_2 + x_3 - x_4 + 4x_5 &= 4 \\
x_1 - x_2 + x_3 - 2x_4 + 3x_5 &= 2 \\
x_2 + x_3 + 2x_5 &= 1 \\
\end{align*}
\]

(a) Compute the general solution of the linear system. Identify the independent variables.

Form the augmented matrix and perform Gauss-Jordan elimination:

\[
[A | b] = \begin{pmatrix}
2 & 1 & 1 & -3 & 9 & 7 \\
1 & 1 & 1 & -1 & 4 & 4 \\
1 & -1 & -1 & -2 & 3 & 2 \\
0 & 1 & 1 & 0 & 2 & 1 \\
\end{pmatrix}
\]

\[
\begin{align*}
\sim & \begin{pmatrix}
1 & 1 & 1 & -1 & 4 & 4 \\
0 & -1 & -1 & 1 & 1 & -2 \\
0 & 1 & 1 & 0 & 2 & 1 \\
\end{pmatrix} \\
\sim & \begin{pmatrix}
1 & 0 & 0 & -2 & 5 & 3 \\
0 & 1 & 1 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & -3 & 0 \\
0 & 0 & 0 & -1 & 3 & 0 \\
\end{pmatrix} \\
\sim & \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 3 \\
0 & 1 & 1 & 0 & 2 & 1 \\
0 & 0 & 1 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\end{align*}
\]

This matrix is in reduced echelon form, so \( x_3 \) and \( x_5 \) are free variables and the general solution is

\[
\begin{align*}
x_1 &= x_5 + 3 \\
x_2 &= -x_3 - 2x_5 + 1 \\
x_3 &= x_3 \\
x_4 &= 3x_5 \\
x_5 &= x_5
\end{align*}
\]

(b) Give two different particular solutions to the linear system.

There are many correct solutions. For example, take \( x_3 = 0 \), \( x_5 = 1 \) and \( x_3 = 1 \), \( x_5 = 0 \) to get

\[
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix} &= \begin{bmatrix}
4 \\
-1 \\
0 \\
3 \\
1 \\
\end{bmatrix} \quad \text{and} \quad \\
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix} &= \begin{bmatrix}
3 \\
0 \\
1 \\
0 \\
0 \\
\end{bmatrix}
\end{align*}
\]

(c) Give the vector form of the general solution to the linear system.

\[
\begin{align*}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix} &= \begin{bmatrix}
0 \\
1 \\
1 + x_5 \\
0 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
1 \\
-2 \\
0 \\
3 \\
1 \\
\end{bmatrix} \begin{bmatrix}
x_5 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\end{align*}
\]
Determine whether the following set of vectors is linearly independent or linearly dependent. If the set is linearly dependent, express one of the vectors as a linear combination of the others.

\[ v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 3 \\ -3 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -5 \\ 7 \\ -8 \\ -7 \end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix} 3 \\ -4 \\ 6 \\ 1 \end{bmatrix}. \]

Let \( A = [v_1 \ v_2 \ v_3 \ v_4] \) be the matrix with the \( v \)'s as columns. We need to check if the equation \( Ax = 0 \) has nontrivial solutions.

Form the augmented matrix and perform Gauss-Jordan elimination:

\[
\begin{bmatrix} A | 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -5 & 3 & 0 \\ -1 & 3 & 7 & -4 & 0 \\ 2 & -3 & -8 & 6 & 0 \\ 1 & -3 & -7 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -5 & 3 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & -1 & -2 & -2 & 0 \end{bmatrix} \quad (R_2 + R_1; \ R_3 - 2R_1; \ R_4 - R_1)
\]

\[
\sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix} \quad (R_1 + 2R_2; \ R_3 - R_2; \ R_4 + R_2)
\]

\[
\sim \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (R_1 - R_3; \ R_2 + R_3; \ R_4 + 3R_3)
\]

This matrix is in reduced echelon form, so \( x_3 \) is a free variable and the general solution is

\[
x_1 = x_3 \\
x_2 = -2x_3 \\
x_3 = x_3 \\
x_4 = 0
\]

There are infinitely many solutions, so the set of vectors is linearly dependent. Any non-trivial particular solution gives a dependency between the \( v \)'s.

For example, take \( x_3 = 1 \) to get \( x_1 = 1, \ x_2 = -2 \). Then \( v_1 - 2v_2 + v_3 = 0 \) or

\[ v_1 = 2v_2 - v_3 \]
Define the matrix
\[ B = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix}. \]

(a) Is \( BB^T \) a nonsingular matrix? Explain.

\[ BB^T = \begin{pmatrix} 21 & 10 \\ 10 & 5 \end{pmatrix}. \]

By inspection, the columns of \( BB^T \) are linearly independent. Thus \( BB^T \) is nonsingular.

(b) Is \( B^T B \) a nonsingular matrix? Explain.

\[ B^T B = \begin{pmatrix} 1 & 4 & 2 \\ 4 & 20 & 10 \\ 2 & 10 & 5 \end{pmatrix}. \]

Note that column 2 of \( B^T B \) is twice column 3. Thus the columns are linearly dependent. It follows that \( B^T B \) is singular.
(4) Suppose that $A$ and $B$ are square matrices of the same size. Show that if $A$ is singular but $B$ is not, then $AB$ is singular.

Since $A$ is singular, we know there exists a non-zero vector $v$ with $Av = 0$.

Since $B$ is nonsingular, we know that the equation $Bx = v$ has a (unique) solution. Call this solution $u$.

Note that $u \neq 0$ since $v \neq 0$.

Then

$$
(AB)u = A(Bu)
$$

$$
= Av
$$

$$
= 0
$$

This proves that $AB$ is singular.