(1) Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be given by $f(a, b) = a^2 + b^2$. Consider the statement "if $f(a, b)$ is odd then a is odd or b is odd.

(a) Give the converse of the original statement. Is the converse true? If so, prove it. If not, give a counterexample.

If a is odd or b is odd then $f(a, b)$ is odd.

This is False. If $a = 5$ and $b = -7$ then a and b are both odd but $f(5, -7) = 74$ is even.

(b) Give the contrapositive of the original statement. Is the contrapositive true? If so, prove it. If not, give a counterexample.

If a is even and b is even then $f(a, b)$ is even.

This is True.
Assume a and b are even. Choose $k, \ell \in \mathbb{Z}$ so that $a = 2k$ and $b = 2\ell$. Then

$$\begin{aligned} f(a,b) &= a^2 + b^2 \\ &= (2k)^2 + 2\ell)^2 \\ &= 4k^2 + 4\ell^2 \\ &= 2 \cdot (2k^2 + 2\ell^2) \end{aligned}$$

Thus $f(a, b)$ is even. $\square$

(2) Prove that $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{1}{3}n(n+1)(n+2)$ for every positive integer n.

We prove this by induction on n.

Base Case: $\displaystyle\sum_{i=1}^{1} i(i+1) = 1 \cdot (1+1) = 2$ *and* $\displaystyle\frac{1}{3} \cdot 1 \cdot (1+1) \cdot (1+2) = 2$

Induction Hypothesis: Fix $n \geq 1$ and assume $\displaystyle\sum_{i=1}^{n} i(i+1) = \frac{1}{3}n(n+1)(n+2)$.

Induction Step:

$$\begin{aligned}
\sum_{i=1}^{n+1} i(i+1) &= \left[\sum_{i=1}^{n} i(i+1)\right] + (n+1)(n+2) \\
&= \frac{1}{3}n(n+1)(n+2) + (n+1)(n+2) \quad \textit{by the Induction Hypothesis} \\
&= (n+1)(n+2) \cdot \left[\frac{1}{3}n + 1\right] \\
&= \frac{1}{3}(n+1)(n+2)(n+3)
\end{aligned}$$

Thus the result follows by induction on n. $\square$

(3) Give a careful proof of the following set theoretic identity.

$$C \setminus (B \setminus A) = (A \cap C) \cup (C \setminus B)$$

Do not use Venn diagrams.

Note that $x \notin (B \setminus A)$ *is equivalent to* $x \notin B$ *or* $x \in A$. *(*)*
because $\neg(x \in B \wedge x \notin A) = x \notin B \vee x \in A$

First we prove that $C \setminus (B \setminus A) \subseteq (A \cap C) \cup (C \setminus B)$.

Let $x \in C \setminus (B \setminus A)$. *Then* $x \in C$ *and* $x \notin (B \setminus A)$.
Thus $x \in C$ *and either* $x \notin B$ *or* $x \in A$, *by (*).*
There are 2 cases to check.

Case 1: Assume that $x \notin B$.
Since $x \in C$ *we have* $x \in C \setminus B$.
Thus $x \in (A \cap C) \cup (C \setminus B)$.

Case 2: Assume that $x \in A$.
Since $x \in C$ *we get* $x \in A \cap C$.
Thus $x \in (A \cap C) \cup (C \setminus B)$.

Hence in each case we have shown that $x \in (A \cap C) \cup (C \setminus B)$.
It follows that $C \setminus (B \setminus A) \subseteq (A \cap C) \cup (C \setminus B)$.

Now we prove that $(A \cap C) \cup (C \setminus B) \subseteq C \setminus (B \setminus A)$.

Let $x \in (A \cap C) \cup (C \setminus B)$.
Then $x \in A \cap C$ *or* $x \in C \setminus B$.
There are 2 cases to check.

Case 1: Assume that $x \in A \cap C$.
Then $x \in C$ *and* $x \in A$.
Since $x \in A$ *we get* $x \notin B \setminus A$, *by (*).*
Thus $x \in C \setminus (B \setminus A)$.

Case 2: Assume that $x \in C \setminus B$.
Then $x \in C$ *and* $x \notin B$.
Since $x \notin B$ *we conclude* $x \notin B \setminus A$, *by (*).*
Since $x \in C$ *we get* $x \in C \setminus (B \setminus A)$.

Hence in each case we have shown that $x \in C \setminus (B \setminus A)$.
It follows that $(A \cap C) \cup (C \setminus B) \subseteq C \setminus (B \setminus A)$.

We have proven that the two sets are equal. $\square$

(4) Suppose that functions $f : X \to Y$ and $g : Y \to X$ satisfy $g \circ f = I_X$. Prove that f is an injection and g is a surjection. Give an example to show that $f \circ g$ need not equal I_Y .

Step 1: $f : X \to Y$ is an injection.

Choose any $x_1, x_2 \in X$ and assume that $f(x_1) = f(x_2)$.

$$\begin{aligned} f(x_1) &= f(x_2) \\ g(f(x_1)) &= g(f(x_2)) \\ g \circ f(x_1) &= g \circ f(x_2) \\ I_X(x_1) &= I_X(x_2) \\ x_1 &= x_2 \end{aligned}$$

Thus f is injective.

Step 2: $g : Y \to X$ is a surjection.

Choose any $x \in X$. Define $y = f(x)$.

$$\begin{aligned} g(y) &= g(f(x)) \\ &= g \circ f(x) \\ &= I_X(x) \\ &= x \end{aligned}$$

Thus g is surjective.

One possible example is to take $f(x) = \sqrt{x}$ and $g(x) = x^2$.
Then $X = \mathbb{R}_{\geq 0}$, the non-negative real numbers, and $Y = \mathbb{R}$.
In this case, $g \circ f(x) = x$ but $f \circ g(x) = \sqrt{x^2} = |x|$.