(1) Let \( f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \) be given by \( f(a, b) = a^2 + b^2 \). Consider the statement “if \( f(a, b) \) is odd then \( a \) is odd or \( b \) is odd.

(a) Give the converse of the original statement. Is the converse true? If so, prove it. If not, give a counterexample.

\[
\text{If } a \text{ is odd or } b \text{ is odd then } f(a, b) \text{ is odd.}
\]

\text{This is False. If } a = 5 \text{ and } b = -7 \text{ then } a \text{ and } b \text{ are both odd but } f(5, -7) = 74 \text{ is even.}

(b) Give the contrapositive of the original statement. Is the contrapositive true? If so, prove it. If not, give a counterexample.

\[
\text{If } a \text{ is even and } b \text{ is even then } f(a, b) \text{ is even.}
\]

\text{This is True. Assume } a \text{ and } b \text{ are even. Choose } k, \ell \in \mathbb{Z} \text{ so that } a = 2k \text{ and } b = 2\ell. \text{ Then}
\[
f(a, b) = a^2 + b^2
= (2k)^2 + (2\ell)^2
= 4k^2 + 4\ell^2
= 2 \cdot (2k^2 + 2\ell^2)
\]

\text{Thus } f(a, b) \text{ is even.} \quad \Box
(2) Prove that \[ \sum_{i=1}^{n} i(i+1) = \frac{1}{3} n(n+1)(n+2) \] for every positive integer \( n \).

We prove this by induction on \( n \).

Base Case: \[ \sum_{i=1}^{1} i(i+1) = 1 \cdot (1+1) = 2 \] and \[ \frac{1}{3} \cdot 1 \cdot (1+1) \cdot (1+2) = 2 \]

Induction Hypothesis: Fix \( n \geq 1 \) and assume \[ \sum_{i=1}^{n} i(i+1) = \frac{1}{3} n(n+1)(n+2) \].

Induction Step:

\[
\sum_{i=1}^{n+1} i(i+1) = \left[ \sum_{i=1}^{n} i(i+1) \right] + (n+1)(n+2)
\]
\[
= \frac{1}{3} n(n+1)(n+2) + (n+1)(n+2) \quad \text{by the Induction Hypothesis}
\]
\[
= (n+1)(n+2) + \frac{1}{3} n(n+1)
\]
\[
= \frac{1}{3} (n+1)(n+2)(n+3)
\]

Thus the result follows by induction on \( n \). \( \square \)
(3) Give a careful proof of the following set theoretic identity.

\[ C \setminus (B \setminus A) = (A \cap C) \cup (C \setminus B) \]

Do not use Venn diagrams.

Note that \( x \notin (B \setminus A) \) is equivalent to \( x \notin B \) or \( x \in A \). \((\ast)\)

because \( \neg(x \in B \land x \notin A) = x \notin B \lor x \in A \)

First we prove that \( C \setminus (B \setminus A) \subseteq (A \cap C) \cup (C \setminus B) \).

Let \( x \in C \setminus (B \setminus A) \). Then \( x \in C \) and \( x \notin (B \setminus A) \).

Thus \( x \in C \) and either \( x \notin B \) or \( x \in A \), by \((\ast)\).

There are 2 cases to check.

Case 1: Assume that \( x \notin B \).
Since \( x \in C \) we have \( x \in C \setminus B \).
Thus \( x \in (A \cap C) \cup (C \setminus B) \).

Case 2: Assume that \( x \in A \).
Since \( x \in C \) we get \( x \in A \cap C \).
Thus \( x \in (A \cap C) \cup (C \setminus B) \).

Hence in each case we have shown that \( x \in (A \cap C) \cup (C \setminus B) \).

It follows that \( C \setminus (B \setminus A) \subseteq (A \cap C) \cup (C \setminus B) \).

Now we prove that \( (A \cap C) \cup (C \setminus B) \subseteq C \setminus (B \setminus A) \).

Let \( x \in (A \cap C) \cup (C \setminus B) \).
Then \( x \in A \cap C \) or \( x \in C \setminus B \).

There are 2 cases to check.

Case 1: Assume that \( x \in A \cap C \).
Then \( x \in C \) and \( x \in A \).
Since \( x \in A \) we get \( x \notin B \setminus A \), by \((\ast)\).
Thus \( x \in C \setminus (B \setminus A) \).

Case 2: Assume that \( x \in C \setminus B \).
Then \( x \in C \) and \( x \notin B \).
Since \( x \notin B \) we conclude \( x \notin B \setminus A \), by \((\ast)\).
Since \( x \in C \) we get \( x \in C \setminus (B \setminus A) \).

Hence in each case we have shown that \( x \in C \setminus (B \setminus A) \).

It follows that \( (A \cap C) \cup (C \setminus B) \subseteq C \setminus (B \setminus A) \).

We have proven that the two sets are equal. \( \square \)
Suppose that functions $f : X \to Y$ and $g : Y \to X$ satisfy $g \circ f = I_X$. Prove that $f$ is an injection and $g$ is a surjection. Give an example to show that $f \circ g$ need not equal $I_Y$.

**Step 1:** $f : X \to Y$ is an injection.

Choose any $x_1, x_2 \in X$ and assume that $f(x_1) = f(x_2)$.

\[
\begin{align*}
  f(x_1) &= f(x_2) \\
  g(f(x_1)) &= g(f(x_2)) \\
  g \circ f(x_1) &= g \circ f(x_2) \\
  I_X(x_1) &= I_X(x_2) \\
  x_1 &= x_2
\end{align*}
\]

Thus $f$ is injective.

**Step 2:** $g : Y \to X$ is a surjection.

Choose any $x \in X$. Define $y = f(x)$.

\[
\begin{align*}
  g(y) &= g(f(x)) \\
       &= g \circ f(x) \\
       &= I_X(x) \\
       &= x
\end{align*}
\]

Thus $g$ is surjective.

One possible example is to take $f(x) = \sqrt{x}$ and $g(x) = x^2$.

Then $X = \mathbb{R}_{\geq 0}$, the non-negative real numbers, and $Y = \mathbb{R}$.

In this case, $g \circ f(x) = x$ but $f \circ g(x) = \sqrt{x^2} = |x|$. 