Homework 7 **Math 300C Autumn 2015** **Patrick Perkins**

NOTE: In Problems 5 and 6 below, you may assume that the Continuum Hypothesis holds.

1. (a) Prove that if x is a non-zero rational number and y is an irrational number, then the product xy must be irrational.

   (b) Give an example to show that the product of two irrational numbers may be a rational.

2. From Problems III, page 186, Problem 26:
   Prove that each finite decimal number (except zero) can be written as an infinite decimal in two distinct ways. That is, prove that:

   $$a_0.a_1a_2\ldots a_{n1}a_n\overline{0} = a_0.a_1a_2\ldots a_{n1}(a_n - 1)\overline{9}$$

   where $a_0 \in \mathbb{Z}$, $a_i \in \{0, 1, \ldots, 9\}$ for $i > 0$ and $a_n \geq 1$.

   You may use without proof the fact that $1.\overline{0} = 0.\overline{9}$ (which we proved in class).

3. Use Cantors diagonal argument to write a complete formal proof showing that the interval of real numbers $[2, 3]$ is uncountable.

4. Assume that A is uncountable and B is a countable subset of A. Prove that $A \setminus B$ is uncountable.

5. Determine the cardinality of the following sets. Your answer should be either an integer number, or one of $\aleph_0$, $\aleph_1$, $\aleph_2$ etc. Give a brief justification.

   (a) the irrational numbers
   (b) $\mathbb{Q} \times \mathbb{Q}$
   (c) $S = \{\sqrt{n} \mid n \in \mathbb{Q},\ n \geq 0\}$
   (d) $T = \{\sqrt[m]{n} \mid m, n \in \mathbb{N}\}$
   (e) $A = \{n \in \mathbb{Z} \mid 0 \leq n \leq 41\}$
   (f) the power set of the rational numbers, $\mathcal{P}(\mathbb{Q})$
   (g) the complex numbers: $C = \{x + iy \mid x, y \in \mathbb{R},\ i^2 = -1\}$

6. Give examples of sets (other than precisely $\mathbb{N}_{81}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$) with each of the following cardinalities.

   (a) 81
   (b) 0
   (c) $\aleph_1$
   (d) $\aleph_5$