Homework 5 **Math 300C Autumn 2015** **Patrick Perkins**

You may use any of the definitions and the results from Chapters 10- 11 of the textbook, summarized on the Finite Sets Handout. Anything else you claim regarding finite sets or functions between finite sets should be deduced from these results.

1. Let A and B be two non-empty finite sets.
   (a) Prove that if A and B have the same cardinality n, then there exists a bijection $f : A \to B$.
   (b) Conversely, if there exists a bijection $f : A \to B$, prove that the two sets have the same cardinality.

2. (a) Let B be a proper subset of a set A. Prove that if there exists a bijection $g : A \to B$, then the set A must be infinite. (Hint: use contradiction)

   (b) Construct an explicit bijective function from the set of all integers $\mathbb{Z}$ to the set of natural numbers $\mathbb{N}$.

3. From Problems III, page 182, Problem 1.
   Read this one in the book.
   (Justify your answer using the Inclusion-Exclusion Principle)

4. From Problems III, page 184, Problem 12:
   Suppose there exists an injective function $f : \mathbb{Z}^+ \to X$. Prove by contradiction that X must be an infinite set.

   [Use Corollary 11.1.1 noting that, for any $n \geq 1$, f restricts to give an injection $\mathbb{N}_{n+1} \to X$.]

5. From Problems III, page 184, Problem 14:
   Let $n \in \mathbb{Z}^+$. Suppose that A is a subset of $\mathbb{N}_{2n}$ and that $|A| = n + 1$. Prove that A contains a pair of distinct integers a, b such that a divides b.

   (In other words, if you choose any subset of $n + 1$ integers between 1 and $2n$, there must always be two among them such that one divides the other.)

   [Hint: Let $f(x)$ be the greatest odd integer that divides x. Apply the Pigeonhole Principle to this function. Start by figuring out what domain and codomain you need for this function.]

6. From Problems III, page 185, Problem 20.
   Let X be a set of 10 distinct positive integers less than 107. Use the Pigeonhole Principle to prove that there exist two distinct subsets of X with the same sum.

   Can you do this if $|X| = 9$?