Homework 4 **Math 300C Autumn 2015** **Patrick Perkins**

As always, do this first:

- Read chapters 7-9, paying attention to the examples worked in the textbook
- Attempt the following PRACTICE problems:
  - Chapter 7: 7.1-7.2, 7.4 (i, ii, iv, v), 7.6-7.7
  - Chapter 8: 8.1, 8.2, 8.3, 8.5
  - Chapter 9: 9.1, 9.2, 9.4, 9.5, 9.7

Write up and turn in complete solutions to the following:

1. Define the sequence a_n inductively by $a_1 = 1$ and
$$a_{k+1} = \frac{6a_k + 5}{a_k + 2}$$
Prove that (i) $a_n > 0$ and (ii) $a_n < 5$ for all positive integers n.

2. Suppose that $A \subseteq \mathbb{Z}$. Consider the statement "There is a greatest number in the set A".
   (a) Write out the statement in the propositional calculus: entirely in symbols using the quantifiers $\forall$ and $\exists$.
   (b) Write out the negative of the statement in the propositional calculus.
   (c) Give an example of a set A for which the statement is true.
   (d) Give an example of a set A for which the statement is false.

3. Use the formal definition of limit of a sequence (§8.3) to prove that the sequence $a_n = \dfrac{1}{n^2}$ has limit equal to zero (that is, a_n is what the book calls a "null" sequence).

4. Write down the negation of the formal definition for the limit of a sequence being zero (in symbolical notation, without using the symbol for "not"). Use it to prove that the sequence $b_n = n^2$ does not have limit equal to 0.

5. Determine which of the following functions $f_i : \mathbb{R} \to \mathbb{R}$ are injective, which are surjective and which are bijective. Write down an inverse for each bijection. You may use known facts about these functions from Calculus and Precalc.
   (a) $f_1(x) = x^3$
   (b) $f_2(x) = x^3 - x$
   (c) $f_3(x) = e^x$

6. Suppose X, Y and Z are arbitrary non-empty sets, and $f : X \to Y$, $g : Y \to Z$ are two arbitrary functions. For each statement below, decide if the statement is always true or not. If it is always true, prove it from definitions. If it is not always true, give a counterexample.
   (a) If both f and g are surjective, then $g \circ f : X \to Z$ is also surjective.
   (b) If $g \circ f : X \to Z$ is injective, then $g : Y \to Z$ must be injective.
   (c) If $g \circ f : X \to Z$ is injective, then $f : X \to Y$ must be injective.