Homework 2 **Math 300 C Autumn 2015** **Patrick Perkins**

1. **DeMorgan's laws and contrapositives**

   DeMorgan's laws codify the fact that the negation of a statement of the form "A and B" is "not A or not B", and the negation of a statement of the form "A or B" is "not A and not B".

   Use DeMorgan's laws to write useful contrapositives of the following sentences.

   (a) If x and y are real numbers, then $x + y$ is a real number.

   (b) If xy is even, then x is even or y is even.

   (c) If you earned at least 90% in my class, then you got an A.

   (d) If it rains or snows, then I will go for a walk but I will not ride my bike.

   Prove each of the following theorems. Use a Theorem/Proof format for each one.

2. Let x and y be integers. Then $4|x^2 + y^2$ if and only if x and y are both even.

3. Let a, b and c be integers, with $c \neq 0$. Then $a \mid b$ if and only if $ca \mid cb$.

4. Let x and y be real numbers.

   (a) Give a counterexample to show that $x \leq y \Rightarrow x^2 \leq y^2$ is not always true.

   (b) Prove that for all positive real numbers $x \leq y \Rightarrow \sqrt{x} \leq \sqrt{y}$.

5. Prove that if x is a rational number and y is an irrational number, then their sum $x + y$ is an irrational number.
   Recall and use the definitions:

   - A real number x is a **rational** number if $x = \dfrac{a}{b}$ for some integer numbers a and b, where $b \neq 0$.
   - x is **irrational** if it is not a rational number.

6. Prove by contradiction that there does not exist a smallest positive real number.

7. Let n be a positive integer. Use induction to prove that 3 divides $4^n + 5$.

8. Let n be an integer. Prove that if $n \geq 4$ then $n! > 2^n$

9. Let n be a positive integer. Prove that $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.