Solution to Homework #7, Page 178, Problem 12(b)

Let A, B and C be sets. Let R be a relation from A to B. Let S and T be relations from B to C.

Then $(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$.

Proof: First note that $(S \cap T) \circ R$ and $(S \circ R) \cap (T \circ R)$ are both subsets of $A \times C$.

Let (a, c) be an arbitrary element of $(S \cap T) \circ R$. By the definition of composition of relations, there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S \cap T$.

From the definition of intersection, we know that $(b, c) \in S$ and $(b, c) \in T$. Thus $(a, c) \in S \circ R$, because $(a, b) \in R$ and $(b, c) \in S$. Similarly $(a, c) \in T \circ R$.

Hence $(a, c) \in (S \circ R) \cap (T \circ R)$. Since (a, c) was arbitrary, we conclude that $(S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$.

(But note that they are not equal, in general.)

Solution to Homework #7, Page 186, Problem 8

Let A be a set and R a relation on A. Then R is transitive if and only if $R \circ R \subseteq R$.

Proof: ($\rightarrow$) Assume R is transitive and choose $(a, b) \in R \circ R$. By the definition of $R \circ R$, there exists $c \in A$ such that $(a, c) \in R$ and $(c, b) \in R$. But R is transitive, so this implies that $(a, b) \in R$. Since (a, b) was an arbitrary element of $R \circ R$, we conclude that $R \circ R \subseteq R$.

($\leftarrow$) Assume $R \circ R \subseteq R$. Let (a, b) and (b, c) be elements of R. We must show that $(a, c) \in R$. But the existence of b guarantees that $(a, c) \in R \circ R$. Since $R \circ R \subseteq R$, we conclude that $(a, c) \in R$. Thus R is transitive.