Solution to Homework #5, Page 133, Problem 8

Let A and B be sets. Then $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof: ($\rightarrow$) Assume $A \subseteq B$ and let U be any element of $\mathcal{P}(A)$. We wish to prove that $U \in \mathcal{P}(B)$. Since $U \in \mathcal{P}(A)$ we have $U \subseteq A$, by definition of the power set. Thus if x is any element of U, x is an element of A. Since $A \subseteq B$ we also get $x \in B$. It follows that U is a subset of B, since x was arbitrary. Thus $U \in \mathcal{P}(B)$. But U was also arbitrary so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

($\leftarrow$) Assume $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \subseteq A$ we have $A \in \mathcal{P}(A)$, by definition of the power set. Thus $A \in \mathcal{P}(B)$ by our hypothesis. By the definition of $\mathcal{P}(B)$ this means $A \subseteq B$.