1 (8 points) The position of a particle is given by $\mathbf{r}(t) = 4t\,\mathbf{i} + 2t^2\,\mathbf{j} + \ln t\,\mathbf{k}$. Find all points on the path where the velocity is perpendicular to the acceleration.

$$\begin{aligned} \mathbf{r}'(t) &= 4\,\mathbf{i} + 4t\,\mathbf{j} + \frac{1}{t}\,\mathbf{k} \\ \mathbf{r}''(t) &= 4\,\mathbf{j} - \frac{1}{t^2}\,\mathbf{k} \\ 0 &= \mathbf{r}'(t)\cdot\mathbf{r}''(t) \\ &= 16t - \frac{1}{t^3} \\ 0 &= 16t^4 - 1 \\ t &= \pm\frac{1}{2} \end{aligned}$$

But $t = -\frac{1}{2}$ is not in the domain of the function.

Thus the only point is $\mathbf{r}'\left(\frac{1}{2}\right) = \langle 2, \frac{1}{2}, -\ln 2\rangle$.

2 (8 points) Calculate the equation of the tangent plane to the hyperboloid $3x^2+5y^2-z^2=8$ at the point $(2,-1,3)$.

At the point $(2,-1,3)$, z is positive so we can write $z = \sqrt{3x^2+5y^2-8}$.

$$\frac{\partial z}{\partial x} = \left.\frac{3x}{\sqrt{3x^2+5y^2-8}}\right|_{(2,-1)} = 2$$

$$\frac{\partial z}{\partial y} = \left.\frac{5y}{\sqrt{3x^2+5y^2-8}}\right|_{(2,-1)} = -\frac{5}{3}$$

The tangent plane is

$z - 3 = 2(x-2) - \frac{5}{3}(y+1)$

or

$6x - 5y - 3z = 8$

3 (9 points) Find the absolute minimum of the function $f(x, y) = y^2 - xy + x$ on the triangular region in the first quadrant where $x + y \leq 7$.

First calculate the critical points in the region.

$f_x(x, y) = -y + 1$ *so if* $f_x(x, y) = 0$ *then* $y = 1$.

$f_y(x, y) = 2y - x$ *so if* $f_y(x, y) = 0$ *and* $y = 1$ *then* $x = 2$.

The only critical point is $(2, 1)$.

Now consider the boundary. It consists of 3 pieces.

1. $x = 0$ *and* $0 \leq y \leq 7$

Here $f(x, y)$ *restricted to the boundary is* y^2. *This is increasing, so the minimum is at* $y = 0$.

2. $y = 0$ *and* $0 \leq x \leq 7$

Here $f(x, y)$ *restricted to the boundary is* x. *This is increasing, so the minimum is at* $x = 0$.

3. $x = 7 - y$ *and* $0 \leq y \leq 7$

Here $f(x, y)$ *restricted to the boundary is* $2y^2 - 8y + 7$.

Check for critical values in the interval: $4y - 8 = 0$ *so* $y = 2$ *(and* $x = 7 - y = 5$*).*

This is an upward opening parabola, so the minimum is at the critical point.

Thus we need to compute 3 values of $f(x, y)$*:*

$f(2, 1) = 1$

$f(0, 0) = 0$

$f(5, 2) = -1$

The minimum value of $f(x, y)$ *on the region is* -1.

It occurs on the boundary at the point $(5, 2)$.

4 (16 points) Evaluate the following double integrals. Give your answers in exact form.

(a) (8 points) $\displaystyle\int_0^2 \int_{x^2}^4 x^5 e^{y^2}\,dy\,dx$

We first need to reverse the order of integration.

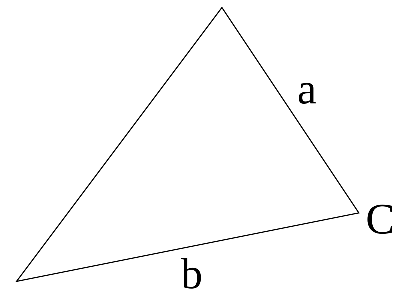
$$
\begin{aligned}
\int_0^4 \int_0^{\sqrt{y}} x^5 e^{y^2}\,dx\,dy &= \int_0^4 \frac{1}{6} x^6 e^{y^2} \Big|_{x=0}^{\sqrt{y}} dy \\
&= \int_0^4 \frac{1}{6} y^3 e^{y^2}\,dy \qquad \textit{let } u = y^2 \textit{ and } du = 2y\,dy \\
&= \int_0^{16} \frac{1}{12} u e^u\,du \\
&= \frac{1}{12}\left(ue^u - e^u\right)\Big|_0^{16} \qquad \textit{Integration by Parts or Guess and Check} \\
&= \frac{1}{12}\left(15e^{16} + 1\right)
\end{aligned}
$$

(b) (8 points) $\displaystyle\int_{-3}^3 \int_0^{\sqrt{9-x^2}} 4x^2 + 5y^3 + 4y^2\,dy\,dx$

First convert to polar coordinates.

$$
\begin{aligned}
\int_0^{\pi} \int_0^3 4r^3 + 5r^4 \sin^3\theta\,dr\,d\theta &= \int_0^{\pi} r^4 + r^5 \sin^3\theta \Big|_{r=0}^{3} d\theta \\
&= \int_0^{\pi} 81 + 243 \sin^3\theta\,d\theta \\
&= 81\pi + \int_0^{\pi} 243\left(1 - \cos^2\theta\right)\sin\theta\,d\theta \qquad \textit{let } u = \cos\theta \\
&= 81\pi + \int_1^{-1} -243\left(1 - u^2\right) du \\
&= 81\pi + 243 \int_{-1}^{1} \left(1 - u^2\right) du \\
&= 81\pi + 243\left(u - u^3/3\right)\Big|_{-1}^{1} \\
&= 81\pi + 324
\end{aligned}
$$

5 (9 points) Clovis must calculate the area of a triangular field. He measures edge **a** to be 150ft and edge **b** to be 200ft. He measures angle **C** to be 60°. The error in his edge measurements is half a foot. His angle measurement has an error of 2°.
Use a linear approximation to estimate the maximum error in his area calculation.
(Recall that the area of a triangle is given by $\frac{1}{2}ab\sin\theta$.)



Recall that you must use radian measure if you are going to use calculus on trigonometric functions.

Thus we have

$a = 150$

$b = 200$

$\Delta a = \Delta b = \frac{1}{2}$

$C = \dfrac{\pi}{3}$

$\Delta C = \dfrac{\pi}{90}$

Let A be the area of the triangle. Then $A = \frac{1}{2}ab\sin\theta$

$$\frac{\partial A}{\partial a} = \frac{1}{2}b\sin\theta\Big|_{(150,200,\pi/3)} = 50\sqrt{3}$$

$$\frac{\partial A}{\partial b} = \frac{1}{2}a\sin\theta\Big|_{(150,200,\pi/3)} = \frac{75\sqrt{3}}{2}$$

$$\frac{\partial A}{\partial C} = \frac{1}{2}ab\cos\theta\Big|_{(150,200,\pi/3)} = 7500$$

Using the linearization $\Delta A \approx \dfrac{\partial A}{\partial a}\Delta a + \dfrac{\partial A}{\partial b}\Delta b + \dfrac{\partial A}{\partial C}\Delta C$ *gives*

$$\begin{aligned}\Delta A &\approx 50\sqrt{3}\cdot\frac{1}{2} + \frac{75\sqrt{3}}{2}\cdot\frac{1}{2} + 7500\cdot\frac{\pi}{90}\\ &= \frac{175}{4}\sqrt{3} + \frac{250}{3}\pi\\ &\approx 337.6 \textit{ square feet}\end{aligned}$$