\begin{itemize}
\item[(1) 8 points] Calculate all the second order partial derivatives of \(g(x, y, z) = \frac{x}{y + 3z}\).
\end{itemize}

There are 6 distinct ones.
First, \(g_x(x, y, z) = \frac{1}{y + 3z}\)
\[g_{xx}(x, y, z) = 0\]
\[g_{xy}(x, y, z) = -\frac{1}{(y + 3z)^2}\]
\[g_{xz}(x, y, z) = -\frac{3}{(y + 3z)^2}\]
Next, \(g_y(x, y, z) = -\frac{x}{(y + 3z)^2}\)
\[g_{yy}(x, y, z) = \frac{2x}{(y + 3z)^3}\]
\[g_{yz}(x, y, z) = \frac{6x}{(y + 3z)^3}\]
Finally, \(g_z(x, y, z) = -\frac{3x}{(y + 3z)^2}\)
\[g_{zz}(x, y, z) = \frac{18x}{(y + 3z)^3}\]

\begin{itemize}
\item[(2) 8 points] Find the linear approximation of the function \(f(x, y) = y \sin(2x - y)\) at \((1, 2)\) and use it to approximate \(f(1.02, 1.9)\).
\end{itemize}

First compute \(f_x(x, y) = 2y \cos(2x - y)\) and \(f_y(x, y) = \sin(2x - y) - y \cos(2x - y)\).
Then, \(f_x(1, 2) = 4, f_y(1, 2) = -2\) and \(f(1, 2) = 0\).
The equation of the tangent plane is \(z = 4(x - 1) - 2(y - 2)\).
The linearization is \(L(x, y) = 4(x - 1) - 2(y - 2)\).
\(f(1.02, 1.9) \approx L(1.02, 1.9) = 0.28\).
Choose variables $x$, $y$ and $z$ with $x, y, z > 0$.

First formulation. Minimize $x^2 + y^2 + z^2$ subject to the constraints $x + y + z = 12$ and $x, y, z > 0$.

Use the constraint to write $z = 12 - x - y$.

Second formulation. Minimize $S(x, y) = x^2 + y^2 + (12 - x - y)^2$ subject to $x, y, z > 0$.

Calculate the critical points.

$S_x(x, y) = 2x - 2(12 - x - y) = 0$ gives $y = 12 - 2x$.

$S_y(x, y) = 2y - 2(12 - x - y) = 0$ gives $x = 12 - 2y$.

Combining we get $x = 12 - 2(12 - 2x)$, or $x = 4$.

Thus $y = 12 - 2x = 4$ and $z = 12 - x - y = 4$.

Now use the Second Derivative Test to verify that $(4, 4)$ gives a minimum for $S(x, y)$.

Calculate the Hessian determinant.

$$H(x, y) = \begin{vmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 12 > 0.$$  

$H(x, y) = 12$ is positive for all values $x, y$ and $S_{xx} = 4$ is also positive. Thus the point $(4, 4)$ gives a local minimum for the function $S(x, y)$. It is the unique critical point and so must give a global minimum.
(a) (8 points) \[ \iint_{R} \frac{x}{1 + xy} \, dA, \quad R = [0, 1] \times [0, 2] \]

By Fubini’s Theorem, we can compute this with the iterated integral

\[
\int_0^1 \int_0^2 \frac{x}{1 + xy} \, dy \, dx = \int_0^1 \ln(1 + xy)^2_{y=0} \, dx
\]

\[
= \int_0^1 \ln(1 + 2x) \, dx \quad \text{integrate by parts} \quad u = \ln(1 + 2x), \quad dv = dx
\]

\[
= x \ln(1 + 2x) \bigg|_0^1 - \int_0^1 \frac{2x}{1 + 2x} \, dx
\]

\[
= \ln 3 - \int_0^1 1 - \frac{1}{1 + 2x} \, dx
\]

\[
= \ln 3 - \left( x - \frac{1}{2} \ln(1 + 2x) \right) \bigg|_0^1
\]

\[
= \ln 3 - 1 + \frac{1}{2} \ln 3
\]

\[
= \frac{3}{2} \ln 3 - 1
\]

(b) (8 points) \[ \iint_{D} xy^2 \, dA, \quad D \text{ is the triangle with vertices } (0,0), (0,2) \text{ and } (1,2). \]

I’ll set it up as an iterated integral with \(dx\) first. You can do it the other way, too.

\[
\int_0^2 \int_0^{\sqrt{y}/2} xy^2 \, dx \, dy = \int_0^2 \frac{1}{2} x^2 y^2 \bigg|_{x=0}^{\sqrt{y}/2} \, dy
\]

\[
= \int_0^2 \frac{1}{8} y^4 \, dy
\]

\[
= \frac{1}{40} y^5 \bigg|_0^2
\]

\[
= \frac{4}{5}
\]
5 (9 points) Let \( \mathbf{r}(t) = e^t \hat{i} + 2e^t \sin t \hat{j} + 2e^t \cos t \hat{k} \). Reparameterize the curve with respect to arclength measured from the point \((1, 0, 2)\).

*First note that the point \((1, 0, 2)\) corresponds to \( t = 0 \). Let \( s \) represent the desired arclength.

We need to compute \( s \) as a function of \( t \).

\[
s = \int_0^t |\mathbf{r}'(u)| \, du \quad \text{(here } u \text{ is a dummy variable)}
\]

\[
= \int_0^t \sqrt{(e^u)^2 + (2e^u \sin u + 2e^u \cos u)^2 + (2e^u \cos u - 2e^u \sin u)^2} \, du
\]

\[
= \int_0^t \sqrt{e^{2u} + 4e^{2u}(\sin^2 u + 2 \sin u \cos u + \cos^2 u + \cos^2 u - 2 \sin u \cos u + \sin^2 u)} \, du
\]

\[
= \int_0^t \sqrt{9e^{2u}} \, du
\]

\[
= \int_0^t 3e^u \, du
\]

\[
= 3(e^t - 1)
\]

Thus \( t = \ln(1 + s/3) \). Plug this into the original formula for \( \mathbf{r}(t) \) to get

\[
\mathbf{r}(s) = \left( 1 + s/3, 2(1 + s/3) \sin \ln(1 + s/3), 2(1 + s/3) \cos \ln(1 + s/3) \right)
\]