Determine whether the series is convergent. Justify your answers.

(a) (6 points) \[ \sum_{n=0}^{\infty} (-1)^n \frac{n}{n^2 + 7} \]

Use the Alternating Series Test:

1. It alternates, \( \frac{n}{n^2 + 7} > 0 \)

2. \( \lim_{n \to \infty} \frac{n}{n^2 + 7} = 0 \)

3. Let \( f(x) = \frac{x}{x^2 + 7} \). Then \( f'(x) = -\frac{x^2 - 7}{(x^2 + 7)^2} < 0 \) for \( x > \sqrt{7} \).

Thus the sequence is decreasing for \( n \geq 3 \).

The series converges by the AST.

(b) (6 points) \[ \sum_{n=0}^{\infty} \frac{(-7)^{n+1}}{2^{3n}} \]

\[ \sum_{n=0}^{\infty} \frac{(-7)^{n+1}}{2^{3n}} = \sum_{n=0}^{\infty} -7 \cdot \frac{(-7)^n}{2^{3n}} = \sum_{n=0}^{\infty} -7 \cdot \frac{(-7)^n}{8^n} \]

This is geometric with \( a = -7 \) and \( r = -7/8 \).

Since \(-1 < r < 1\), it converges to \( \frac{-7}{1 + 7/8} = \frac{-56}{15} \).
(a) (2 points) If \( \{a_n\} \) is a sequence with limit 5 then \( \sum_{n=1}^{\infty} (a_n - 5) \) converges.

*False, consider \( \sum_{n=1}^{\infty} \frac{5n + 1}{n} \).*

(b) (2 points) If \( \sum_{n=0}^{\infty} a_n \) is divergent then \( \sum_{n=0}^{\infty} |a_n| \) is also divergent.

*True, if \( \sum_{n=0}^{\infty} |a_n| \) were convergent, then \( \sum_{n=0}^{\infty} a_n \) would be absolutely convergent.*

(c) (2 points) If \( \sum_{n=1}^{\infty} c_n 3^n \) diverges then \( \sum_{n=1}^{\infty} c_n (-4)^n \) also diverges.

*True, this is saying that \( \sum_{n=1}^{\infty} c_n x^n \) has a radius of convergence that is \( \leq 3 \).*

(d) (2 points) \( \sum_{n=1}^{\infty} n^{-\ln(3)} \) converges.

*True, this is a p-series with \( p = \ln(3) > 1 \).*
Consider the power series \( \sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+3)^2 7^n} \).

(a) (7 points) Find the interval of convergence. Do not check the endpoints.

Use the Ratio Test:

\[
\lim_{n \to \infty} \left| \frac{(x-3)^{n+1}}{(n+4)^2 7^{n+1}} \cdot \frac{(n+3)^2 7^n}{(x-3)^n} \right| = \lim_{n \to \infty} \frac{1}{7} \cdot \left( \frac{n+3}{n+4} \right)^2 |x-3| = \frac{1}{7} |x-3|
\]

Setting \( \frac{1}{7} |x-3| < 1 \) gives the interval \(-4 < x < 10\).

(In fact, this one converges at both endpoints.)

(b) (7 points) How many terms should we use to estimate \( \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+3)^2 7^n} \) to within 0.01?

The series starts with \( \frac{1}{9} - \frac{1}{16 \cdot 7} + \frac{1}{25 \cdot 49} - \cdots \)

Since \( \frac{1}{16 \cdot 7} < \frac{1}{100} \) we need only one term to get the desired degree of accuracy.
4. (8 points) Compute the 3rd-degree Taylor polynomial of \( f(x) = \sqrt{x} \) centered at \( a = -1 \).

\[
\begin{align*}
    f(x) &= x^{1/5} \\
    f'(x) &= \frac{1}{5}x^{-4/5} \\
    f''(x) &= -\frac{4}{25}x^{-9/5} \\
    f'''(x) &= \frac{36}{125}x^{-14/5}
\end{align*}
\]

\[
\begin{align*}
    f(-1) &= -1 \\
    f'(-1) &= \frac{1}{5} \\
    f''(-1) &= \frac{4}{25} \\
    f'''(-1) &= \frac{36}{125}
\end{align*}
\]

\[
\begin{align*}
    f(x) &\approx -1 + \frac{1}{5}(x + 1) + \frac{4/25}{2!}(x + 1)^2 + \frac{36/125}{3!}(x + 1)^3 \\
    &= -1 + \frac{1}{5}(x + 1) + \frac{2}{25}(x + 1)^2 + \frac{6}{125}(x + 1)^3
\end{align*}
\]

5. (8 points) Evaluate the indefinite integral \( \int x^2 \tan^{-1}(2x) \, dx \) as a power series.

Use \( \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+1} \) to get \( x^2 \tan^{-1}(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n + 1} x^{2n+3} \).

Then

\[
\int x^2 \tan^{-1}(2x) \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n + 1} x^{2n+3} \, dx
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n + 1} \int x^{2n+3} \, dx \quad (by \ the \ theorem \ on \ term-by-term \ integration)
\]

\[
= C + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n + 1)(2n + 4)} x^{2n+4}
\]