(a) (6 points) \[ \int \sin^3 \theta \cos^5 \theta \, d\theta \]

\[
\int \sin^3 \theta \cos^5 \theta \, d\theta = \int \sin^2 \theta \cos^5 \theta \sin \theta \, d\theta
\]

\[
= \int \cos^5 \theta (1 - \cos^2 \theta) \sin \theta \, d\theta \quad \text{let } u = \cos \theta
\]

\[
= \int -u^5 (1 - u^2) \, du
\]

\[
= \frac{1}{8} u^8 - \frac{1}{6} u^6 + C
\]

\[
= \frac{1}{8} \cos^8 \theta - \frac{1}{6} \cos^6 \theta + C
\]

(b) (6 points) \[ \int \frac{1}{y \sqrt{y^2 - 16}} \, dy \]

Let \( y = 4 \sec \theta \). Then \( dy = 4 \sec \theta \tan \theta \, d\theta \) and \( \sqrt{y^2 - 16} = 4 \tan \theta \)

\[
\int \frac{1}{y \sqrt{y^2 - 16}} \, dy = \int \frac{1}{4 \sec \theta \cdot 4 \tan \theta} \cdot 4 \sec \theta \tan \theta \, d\theta
\]

\[
= \int \frac{1}{4} \, d\theta
\]

\[
= \frac{1}{4} \theta + C
\]

\[
= \frac{1}{4} \tan^{-1} \frac{1}{4} \sqrt{y^2 - 16} + C
\]
Compute the following definite integrals. Give your answers in exact form.

(a) (6 points) \( \int_{3}^{5} \frac{6x^2}{x^2 - 3x + 2} \, dx \)

First use Partial Fractions to write

\[
\frac{6x^2}{x^2 - 3x + 2} = 6 + \frac{24}{x - 2} - \frac{6}{x - 1}
\]

\[
\int_{3}^{5} \frac{6x^2}{x^2 - 3x + 2} \, dx = \int_{3}^{5} 6 + \frac{24}{x - 2} - \frac{6}{x - 1} \, dx
\]

\[
= 6x + 24 \ln |x - 2| - 6 \ln |x - 1| \bigg|_{3}^{5}
\]

\[
= 12 + 24 \ln 3 - 6 \ln 4 + 6 \ln 2
\]

(b) (6 points) \( \int_{0}^{1} t \sin^{-1} t \, dt \)

First use Integration by Parts with \( u = \sin^{-1} t \) and \( dv = t \, dt \) to get

\[
\int_{0}^{1} t \sin^{-1} t \, dt = \frac{1}{2} t^2 \sin^{-1} t \bigg|_{0}^{1} - \frac{1}{2} \int_{0}^{1} \frac{t^2}{\sqrt{1-t^2}} \, dt
\]

\[
= \frac{\pi}{4} - \frac{1}{2} \int_{0}^{1} \frac{t^2}{\sqrt{1-t^2}} \, dt
\]

To solve \( \int \frac{t^2}{\sqrt{1-t^2}} \, dt \), set \( t = \sin \theta \) so that \( dt = \cos \theta \, d\theta \) and \( \sqrt{1-t^2} = \cos \theta \).

\[
\int \frac{t^2}{\sqrt{1-t^2}} \, dt = \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta \, d\theta
\]

\[
= \int \sin^2 \theta \, d\theta
\]

\[
= \frac{1}{2} \int 1 - \cos 2\theta \, d\theta
\]

\[
= \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta + C
\]

\[
= \frac{1}{2} \theta - \frac{1}{2} \sin \theta \cos \theta + C
\]

\[
= \frac{1}{2} \sin^{-1} t - \frac{1}{2} t \sqrt{1-t^2} + C
\]

Thus

\[
\int_{0}^{1} \frac{t^2}{\sqrt{1-t^2}} \, dt = \frac{1}{2} \sin^{-1} t - \frac{1}{2} t \sqrt{1-t^2} \bigg|_{0}^{1} = \frac{\pi}{4}
\]

and

\[
\int_{0}^{1} t \sin^{-1} t \, dt = \frac{\pi}{4} - \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}
\]
A rope is used to pull a bucket full of water up from a well that is 10 m deep. The rope has a total mass of 6 kg. The bucket of water has a mass of 17 kg. The acceleration due to gravity is $9.8 \, \text{m/sec}^2$. Set up an integral that computes the work done in lifting the bucket all the way up. **Do not simplify or evaluate the integral.**

Choose a coordinate system with $y = 0$ at the top of the well and $y = 10$ at the bottom.

Let $W(y)$ be the weight of the load at depth $y$ meters.

Then the integral we want is

$$\int_0^{10} W(y) \, dy.$$ 

The weight of the bucket is $9.8 \times 17 = 166.6$ Newtons.

The rope has a mass of 0.6 kg/m. When the load is at depth $y$ meters, we are lifting $y$ meters of rope. This weighs $9.8 \times 0.6y = 5.88y$ Newtons.

Thus the total weight of the load is $W(y) = 166.6 + 5.88y$ Newtons

The total work is given by

$$\int_0^{10} 166.6 + 5.88y \, dy.$$ 

---

Use the Trapezoid Rule with $n = 5$ to approximate the average value of the function $\phi(x) = \cos(1/x)$ on the interval $x = 1$ to $x = 4$. Round your answer to 3 decimal places.

We must approximate

$$\frac{1}{4 - 1} \int_1^4 \phi(x) \, dx \quad \text{where} \quad \phi(x) = \sin(1/x).$$

$$\Delta x = \frac{4 - 1}{5} = 0.6$$

$x_0 = 1, x_1 = 1.6, x_2 = 2.2, x_3 = 2.8, x_4 = 3.4, x_5 = 4$

$$\int_1^4 \phi(x) \, dx \approx \frac{1}{2} \cdot \Delta x \cdot [\phi(x_1) + 2\phi(x_2) + 2\phi(x_3) + 2\phi(x_4) + \phi(x_5)]$$

$$= 0.3 \cdot [\phi(1) + 2\phi(1.6) + 2\phi(2.2) + 2\phi(2.8) + 2\phi(3.4) + \phi(4)]$$

$$\approx 0.3 \times 8.715978$$

$$= 2.614793$$

The average value is

$$\frac{1}{3} \int_1^4 \phi(x) \, dx \approx \frac{1}{3} \times 2.614793 \approx 0.872$$
(10 points) Determine if the improper integral \( \int_{-1}^{0} \frac{e^{1/t}}{t^3} \, dt \) is convergent or divergent. If it is convergent, evaluate it.

**Step 1: Calculate \( \int \frac{e^{1/t}}{t^3} \, dt \).**

First make the substitution \( x = \frac{1}{t} \). Then \( dx = -\frac{1}{t^2} \, dt \).

The integral transforms to \( \int -xe^x \, dx \).

Now use Integration by Parts with \( u = -x \) and \( dv = e^x \, dx \) to get

\[
\int -xe^x \, dx = -xe^x + e^x + C = (1 - x)e^x + C
\]

Thus \( \int \frac{e^{1/t}}{t^3} \, dt = \left(1 - \frac{1}{t}\right)e^{1/t} + C \).

**Step 2: Compute \( \lim_{b \to 0^-} \int_{-1}^{b} \frac{e^{1/t}}{t^3} \, dt \).**

\[
\lim_{b \to 0^-} \int_{-1}^{b} \frac{e^{1/t}}{t^3} \, dt = \lim_{b \to 0^-} \left(1 - \frac{1}{b}\right)e^{1/b} \bigg|_{-1}^{b} \quad \text{by Step 1}
\]

\[
= \lim_{b \to 0^-} \left(1 - \frac{1}{b}\right)e^{1/b} - \frac{2}{e}
\]

\[
= \lim_{b \to 0^-} \frac{1 - \frac{1}{b}}{e^{1/b} - \frac{2}{e}} \quad \text{\( \infty \infty \) l'Hôpital's Rule}
\]

\[
= \lim_{b \to 0^-} \frac{1/b^2}{e^{1/b} - \frac{2}{e}} - \frac{2}{e}
\]

\[
= \lim_{b \to 0^-} e^{1/b} - \frac{2}{e}
\]

\[
= \frac{2}{e}
\]

\[
= -\frac{2}{e}
\]