(12 points) Compute the following indefinite integrals.

(a) (6 points) \[ \int \sin^5 \theta \cos^3 \theta \, d\theta \]

\[
\begin{align*}
\int \sin^5 \theta \cos^3 \theta \, d\theta &= \int \sin^5 \theta \cos^2 \theta \cos \theta \, d\theta \\
&= \int \sin^5 \theta (1 - \sin^2 \theta) \cos \theta \, d\theta \quad \text{let } u = \sin \theta \\
&= \int u^5 (1 - u^2) \, du \\
&= \frac{1}{6} u^6 - \frac{1}{8} u^8 + C \\
&= \frac{1}{6} \sin^6 \theta - \frac{1}{8} \sin^8 \theta + C
\end{align*}
\]

(b) (6 points) \[ \int \frac{1}{y\sqrt{y^2 - 25}} \, dy \]

Let \( y = 5 \sec \theta \). Then \( dy = 5 \sec \theta \tan \theta \, d\theta \) and \( \sqrt{y^2 - 25} = 5 \tan \theta \)

\[
\begin{align*}
\int \frac{1}{y\sqrt{y^2 - 25}} \, dy &= \int \frac{1}{5 \sec \theta \cdot 5 \tan \theta} \, 5 \sec \theta \tan \theta \, d\theta \\
&= \int \frac{1}{5} \, d\theta \\
&= \frac{1}{5} \theta + C \\
&= \frac{1}{5} \tan^{-1} \left( \frac{1}{5} \sqrt{y^2 - 25} \right) + C
\end{align*}
\]
2 (12 points) Compute the following definite integrals. Give your answers in exact form.

(a) (6 points) \[ \int_{3}^{5} \frac{5x^2}{x^2 - 3x + 2} \, dx \]

*First use Partial Fractions to write*

\[ \frac{5x^2}{x^2 - 3x + 2} = 5 + \frac{20}{x - 2} - \frac{5}{x - 1} \]

\[ \int_{3}^{5} \frac{5x^2}{x^2 - 3x + 2} \, dx = \int_{3}^{5} 5 + \frac{20}{x - 2} - \frac{5}{x - 1} \, dx \]

\[ = 5x + 20 \ln |x - 2| - 5 \ln |x - 1| \bigg|_{3}^{5} \]

\[ = 10 + 20 \ln 3 - 5 \ln 4 + 5 \ln 2 \]

\[ = 10 + 20 \ln 3 - 5 \ln 2 \]

(b) (6 points) \[ \int_{0}^{1} t \sin^{-1} t \, dt \]

*First use Integration by Parts with \( u = \sin^{-1} t \) and \( dv = t \, dt \) to get*

\[ \int_{0}^{1} t \sin^{-1} t \, dt = \frac{1}{2} t^2 \sin^{-1} t \bigg|_{0}^{1} - \frac{1}{2} \int_{0}^{1} \frac{t^2}{\sqrt{1 - t^2}} \, dt \]

\[ = \frac{\pi}{4} - \frac{1}{2} \int_{0}^{1} \frac{t^2}{\sqrt{1 - t^2}} \, dt \]

*To solve \( \int \frac{t^2}{\sqrt{1 - t^2}} \, dt \), set \( t = \sin \theta \) so that \( dt = \cos \theta \, d\theta \) and \( \sqrt{1 - t^2} = \cos \theta \).*

\[ \int \frac{t^2}{\sqrt{1 - t^2}} \, dt = \int \frac{\sin^2 \theta}{\cos \theta} \, \cos \theta \, d\theta \]

\[ = \int \sin^2 \theta \, d\theta \]

\[ = \frac{1}{2} \int 1 - \cos 2\theta \, d\theta \]

\[ = \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta + C \]

\[ = \frac{1}{2} \theta - \frac{1}{2} \sin \theta \cos \theta + C \]

\[ = \frac{1}{2} \left( \sin^{-1} t - \frac{1}{2} t \sqrt{1 - t^2} \right) + C \]

*Thus*

\[ \int_{0}^{1} \frac{t^2}{\sqrt{1 - t^2}} \, dt = \frac{1}{2} \sin^{-1} t - \frac{1}{2} t \sqrt{1 - t^2} \bigg|_{0}^{1} = \frac{\pi}{4} \]

*and*

\[ \int_{0}^{1} t \sin^{-1} t \, dt = \frac{\pi}{4} - \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8} \]
\[ 3 \text{ (8 points)} \quad \text{A rope is used to pull a bucket full of water up from a well that is 10 m deep. The rope has a total mass of 5 kg. The bucket of water has a mass of 11 kg. The acceleration due to gravity is 9.8 m/sec}^2. \text{ Set up an integral that computes the work done in lifting the bucket all the way up. Do not simplify or evaluate the integral.} \]

Choose a coordinate system with \( y = 0 \) at the top of the well and \( y = 10 \) at the bottom.

Let \( W(y) \) be the weight of the load at depth \( y \) meters.

Then the integral we want is \( \int_0^{10} W(y) \, dy \).

The weight of the bucket is \( 9.8 \times 11 = 107.8 \) Newtons.

The rope has a mass of 0.5 kg/m. When the load is at depth \( y \) meters, we are lifting \( y \) meters of rope. This weighs \( 9.8 \times 0.5y = 4.9y \) Newtons.

Thus the total weight of the load is \( W(y) = 107.8 + 4.9y \) Newtons.

The total work is given by \( \int_0^{10} 107.8 + 4.9y \, dy \).

\[ 4 \text{ (8 points)} \quad \text{Use the Trapezoid Rule with } n = 5 \text{ to approximate the average value of the function } \phi(x) = \sin(1/x) \text{ on the interval } x = 1 \text{ to } x = 4. \text{ Round your answer to 3 decimal places.} \]

We must approximate \( \frac{1}{4 - 1} \int_1^4 \phi(x) \, dx \) where \( \phi(x) = \sin(1/x) \).

\[ \Delta x = \frac{4 - 1}{5} = 0.6 \]

\[ x_0 = 1, \quad x_1 = 1.6, \quad x_2 = 2.2, \quad x_3 = 2.8, \quad x_4 = 3.4, \quad x_5 = 4 \]

\[
\begin{align*}
\int_1^4 \phi(x) \, dx & \approx \frac{1}{2} \cdot \Delta x \cdot [\phi(x_1) + 2\phi(x_2) + 2\phi(x_3) + 2\phi(x_4) + \phi(x_5)] \\
& = 0.3 \cdot [\phi(1) + 2\phi(1.6) + 2\phi(2.2) + 2\phi(2.8) + 2\phi(3.4) + \phi(4)] \\
& \approx 0.3 \times 4.416166 \\
& = 1.32485
\end{align*}
\]

The average value is \( \frac{1}{3} \int_1^4 \phi(x) \, dx \approx \frac{1}{3} \times 1.32485 \approx 0.442 \)
Determine if the improper integral \( \int_{-1}^{0} \frac{e^{1/t}}{t^3} dt \) is convergent or divergent. If it is convergent, evaluate it.

**Step 1:** Calculate \( \int \frac{e^{1/t}}{t^3} dt \).

First make the substitution \( x = \frac{1}{t} \). Then \( dx = -\frac{1}{t^2} dt \).

The integral transforms to \( \int -xe^x dx \).

Now use Integration by Parts with \( u = -x \) and \( dv = e^x dx \) to get

\[
\int -xe^x dx = -xe^x + e^x + C = (1 - x)e^x + C
\]

Thus \( \int \frac{e^{1/t}}{t^3} dt = \left(1 - \frac{1}{t}\right)e^{1/t} + C \).

**Step 2:** Compute \( \lim_{b \to 0^-} \int_{-1}^{b} \frac{e^{1/t}}{t^3} dt \).

\[
\lim_{b \to 0^-} \int_{-1}^{b} \frac{e^{1/t}}{t^3} dt = \lim_{b \to 0^-} \left(1 - \frac{1}{b}\right)e^{1/b} \bigg|_{-1}^{b} \quad \text{by Step 1}
\]

\[
= \lim_{b \to 0^-} \left(1 - \frac{1}{b}\right)e^{1/b} - \frac{2}{e}
\]

\[
= \lim_{b \to 0^-} \frac{1 - \frac{1}{b}}{e^{-1/b}} - \frac{2}{e} \quad \text{\( \sim \) } l'Hôpital's Rule
\]

\[
= \lim_{b \to 0^-} \frac{1/b^2}{1/b^2 \cdot e^{-1/b}} - \frac{2}{e}
\]

\[
= \lim_{b \to 0^-} e^{1/b} - \frac{2}{e}
\]

\[
= \frac{-2}{e}
\]