Math 125, Winter 2001
Final Exam Solutions

1. (a) Find the derivative of \( f(x) = 10^{\ln x} \).

\[ \text{Solution 1: Use logarithmic differentiation. Let } y = f(x). \]
\[ \ln y = \ln (10^{\ln x}) = (\ln x)(\ln 10) \]
\[ \frac{y'}{y} = \frac{\ln 10}{x} \]
\[ y' = y \frac{\ln 10}{x} = 10^{\ln x} \frac{\ln 10}{x} \]

\[ \text{Solution 2: } f(x) = e^{(\ln x)(\ln 10)} \]
\[ f'(x) = e^{(\ln x)(\ln 10)} \frac{(\ln 10)}{x} \]
\[ = 10^{\ln x} \frac{\ln 10}{x} \]

\text{Solution 3: } f(x) = e^{(\ln x)(\ln 10)} = x^{\ln 10}
\[ f'(x) = (\ln 10) x^{(\ln 10) - 1} \]

1. (b) Find the indefinite integral \( \int \frac{(x^2 - 1)^{3/2}}{x} \, dx \).

(You may assume \( x \geq 1 \).)

\text{Solution: Inverse trigonometric substitution.}
\[ \theta = \sec^{-1} x \]
\[ x = \sec \theta \]
\[ dx = \sec \theta \tan \theta \, d\theta \]

\text{Note: Since } x \geq 1, 0 \leq \theta < \pi/2, \text{ so tan } \theta \geq 0.

\[ \int \frac{(x^2 - 1)^{3/2}}{x} \, dx = \int \frac{(\sec^2 \theta - 1)^{3/2}}{\sec \theta} \sec \theta \tan \theta \, d\theta \]
\[ = \int (\tan^2 \theta)^{3/2} \tan \theta \, d\theta \]
\[ = \int \tan^4 \theta \, d\theta \]
\[ = \int \tan^2 \theta (\sec^2 \theta - 1) \, d\theta \]
\[ = \int \tan^2 \theta \sec^2 \theta - \tan^2 \theta \, d\theta \]
\[ = \int \tan^2 \theta \sec^2 \theta - \sec^2 \theta + 1 \, d\theta \]
\[ = \frac{1}{3} \tan^3 \theta - \tan \theta + \theta + C \]

Using the reference triangle, tan \( \theta = \sqrt{x^2 - 1} \), so
\[ \int \frac{(x^2 - 1)^{3/2}}{x} \, dx = \frac{1}{3} \tan^3 \theta - \tan \theta + \theta + C \]
\[ = \frac{1}{3} (x^2 - 1)^{3/2} - \sqrt{x^2 - 1 + \sec^{-1} x + C} \]

2. (a) Find the average value of \( f(x) = x^2 + x^2 \) on the interval \( 0 \leq x \leq \sqrt{3} \).

\[ \text{Solution: } \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{1}{\sqrt{3}} \int_0^\sqrt{3} x \sqrt{1 + x^2} \, dx \]
\[ = \frac{1}{\sqrt{3}} \left[ \frac{1}{3} (1 + x^2)^{3/2} \right]_0^{\sqrt{3}} \]
\[ = \frac{1}{3\sqrt{3}} \left( 4^{3/2} - 1^{3/2} \right) \]
\[ = \frac{7}{3\sqrt{3}} \]

2. (b) Evaluate the definite integral \( \int_1^4 \frac{\sqrt{x}}{x + 3\sqrt{x} + 2} \, dx \).

\[ \text{Solution: Start with a substitution.} \]
\[ u = \sqrt{x} \]
\[ x = u^2 \]
\[ dx = 2u \, du \]
\[ x = 4 \leftrightarrow u = 2 \]
\[ x = 1 \leftrightarrow u = 1 \]
\[ \int_1^4 \frac{u}{u^2 + 3u + 2} \, du = \int_1^2 \frac{u}{u^2 + 3u + 2} \, du \]
\[ = \int_1^2 \frac{2u^2 + 6u + 4}{2u^2 + 6u + 4} \, du \]
\[ = \int_1^2 \frac{2u^2 + 6u + 4}{-6u - 4} \, du \]

This is now a rational function. Using long division,
\[ \frac{2}{u^2 + 2u + 2} \]
we get \( 2u^2 = (2)(u^2 + 3u + 2) - 6u - 4 \).
\[ \int_1^4 \frac{\sqrt{x}}{x + 3\sqrt{x} + 2} \, dx = \int_1^2 \frac{2(u^2 + 3u + 2) - 6u - 4}{u^2 + 3u + 2} \, du \]
\[ = \int_1^2 \frac{2 - 6u - 4}{u^2 + 3u + 2} \, du \]

The denominator factors: \( u^2 + 3u + 2 = (u + 2)(u + 1) \).

Use partial fractions.
\[ \frac{-6u - 4}{(u + 2)(u + 1)} = \frac{A}{u + 2} + \frac{B}{u + 1} \]
\[ -6u - 4 = A(u + 1) + B(u + 2) \]
Evaluating at \( u = -2 \) gives \( 8 = -A \), so \( A = -8 \). Evaluating at \( u = -1 \) gives \( 2 = B \).

(Alternatively, we could equate the coefficients of the same powers of \( u \) on both sides of the equation:

Coefficient of \( u^0 \): \(-4 = A + 2B\),
Coefficient of \( u^1 \): \(-6 = A + B\);

subtracting the second equation from the first equation gives \( B = 2 \), and then either equation gives \( A = -8 \).

\[
\int_1^4 \frac{\sqrt{x}}{x} + 3\sqrt{x} + 2 \, dx = \int_1^2 \frac{-8}{u + 2} + 2 \frac{u}{u + 1} \, du \\
= [2u - 8\ln |u + 2| + 2\ln |u + 1||]_1^2 \\
= (4 - 8\ln 4 + 2\ln 3) - (2 - 8\ln 3 + 2\ln 2) \\
= 2 - 18\ln 2 + 10\ln 3
\]

3. The region above the curve \( y = \sin x \), below the line \( y = 1 \), and between the lines \( x = 0 \) and \( x = \pi/2 \), is rotated around the line \( x = -1 \). (Read that carefully: this region is rotated around the vertical line \( x = -1 \).)

![Diagram of region](image)

3. (a) Express the volume of the solid of revolution as a definite integral with respect to \( x \). IN THIS PART, DO NOT EVALUATE THE INTEGRAL YET.

**Solution:** Use cylindrical shells.

\[
V = \int_a^b 2\pi rh \, dx,
\]

where \( r \) is the radius of the base of the shell and \( h \) is the height of the shell at \( x \). Because we are rotating around the line \( x = -1 \), \( r = x + 1 \). The curve on the top side of the region is \( y = f_2(x) = 1 \), and the curve on the bottom side of the region is \( y = f_1(x) = \sin x \) (for \( 0 \leq x \leq \pi/2 \)), so \( h = f_2(x) - f_1(x) = 1 - \sin x \).

\[
V = \int_0^{\pi/2} 2\pi(x + 1)(1 - \sin x) \, dx
\]

3. (b) Express the volume of the solid of revolution as a definite integral with respect to \( y \). IN THIS PART, DO NOT EVALUATE THE INTEGRAL YET.

**Solution:** Use washers.

\[
V = \int_c^d \pi R^2 - \pi r^2 \, dy,
\]

where \( R \) is the outer radius and \( r \) is the inner radius of the washer at \( y \). Solving for \( x \) as a function of \( y \), the curve on the right side of the region is \( x = g_2(y) = \sin^{-1} y \), and the curve on the left side of the region is \( x = g_1(y) = 0 \) (for \( 0 \leq y \leq 1 \)). Because we are rotating around the line \( x = -1 \), here

\[
R = g_2(y) + 1 = \sin^{-1} y + 1, \quad \text{and} \quad r = g_1(y) + 1 = 1.
\]

\[
V = \int_0^1 \pi(\sin^{-1} y + 1)^2 - \pi \cdot 1^2 \, dy
\]

3. (c) Find the volume of the solid of revolution.

(Evaluate either integral: your choice.)

**Solution a:**

\[
V = 2\pi \int_0^{\pi/2} (x + 1)(1 - \sin x) \, dx
\]

Find \( \int x \sin x \, dx \) by integration by parts.

\[
u = x, \quad dv = \sin x \, dx
\]

\[
du = dx, \quad v = -\cos x
\]

\[
\int x \sin x \, dx = -x \cos x - \int (-\cos x) \, dx
\]

\[
= -x \cos x + \sin x + C
\]

\[
V = 2\pi \int_0^{\pi/2} (x + 1 - x \sin x - \sin x) \, dx
\]

\[
= 2\pi \left[ \left( \frac{x^2}{2} + x - (-x \cos x + \sin x) + \cos x \right) \right]_0^{\pi/2}
\]

\[
= 2\pi \left[ \left( \frac{\pi^2}{8} + \frac{\pi}{2} - 1 + 0 \right) - (0 + 1) \right]
\]

\[
= \frac{\pi^3}{4} + \pi^2 - 4\pi
\]

**Solution b:**

\[
V = \int_0^1 \pi(\sin^{-1} y + 1)^2 - \pi \cdot 1^2 \, dy
\]

Find \( \int \sin^{-1} y \, dy \) by integration by parts.

\[
u = \sin^{-1} y, \quad dv = dy
\]

\[
du = \frac{1}{\sqrt{1 - y^2}} \quad v = y
\]

\[
\int \sin^{-1} y \, dy = y \sin^{-1} y - \int \frac{y}{\sqrt{1 - y^2}} \, dy
\]

\[
= y \sin^{-1} y + \sqrt{1 - y^2} + C
\]
Also find \( \int (\sin^{-1}y)^2 \, dy \) by integration by parts.

\[
\begin{align*}
    u &= (\sin^{-1}y)^2, & dv &= dy \\
    du &= \frac{2\sin^{-1}y}{\sqrt{1-y^2}}, & v &= y \\
    \int (\sin^{-1}y)^2 \, dy &= y(\sin^{-1}y)^2 - \int \frac{2y\sin^{-1}y}{\sqrt{1-y^2}} 
\end{align*}
\]

This last integral requires one more integration by parts.

\[
\begin{align*}
    u &= \sin^{-1}y, & dv &= \frac{-2y}{\sqrt{1-y^2}} \, dy \\
    du &= \frac{1}{\sqrt{1-y^2}}, & v &= 2\sqrt{1-y^2} \\
    \int \frac{-2y\sin^{-1}y}{\sqrt{1-y^2}} \, dy &= (2\sin^{-1}y)\sqrt{1-y^2} - \int (2) \, dy \\
    &= (2\sin^{-1}y)\sqrt{1-y^2} - 2y + C 
\end{align*}
\]

Assembling the pieces,

\[
\int (\sin^{-1}y)^2 \, dy = y(\sin^{-1}y)^2 + (2\sin^{-1}y)\sqrt{1-y^2} - 2y + C.
\]

**Solution:**

\[
\begin{align*}
    A &= \int_a^b f(x) \, dx \\
    &= \int_0^1 \frac{1}{\sqrt{4-x^2}} \, dx \\
    &= \left[ \sin^{-1}(x/2) \right]_0^1 \\
    &= \sin^{-1}(1/2) - \sin^{-1}(0) = \pi/6 
\end{align*}
\]

4. (b) Find the x-coordinate \( \bar{x} \) of the center of mass of this region.

**Solution:**

\[
\begin{align*}
    \bar{x} &= \frac{1}{A} \int_a^b x f(x) \, dx \\
    &= \frac{6}{\pi} \int_0^1 \frac{x}{\sqrt{4-x^2}} \, dx \\
    &= \frac{6}{\pi} \left[ -\sqrt{4-x^2} \right]_0^1 \\
    &= \frac{6}{\pi} \left[ -\sqrt{3} + \sqrt{3} \right] = \frac{12 - 6\sqrt{3}}{\pi} 
\end{align*}
\]

4. (c) Find the y-coordinate \( \bar{y} \) of the center of mass of this region.

**Solution:**

\[
\begin{align*}
    \bar{y} &= \frac{1}{A} \int_a^b \frac{1}{2} f(x)^2 \, dx \\
    &= \frac{3}{\pi} \int_0^1 \frac{1}{4-x^2} \, dx
\end{align*}
\]

The integrand is now a rational function. The denominator factors: \( 4-x^2 = -(x^2-4) = -(x+2)(x-2) \). Use partial fractions.

\[
\frac{-1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2} \\n-1 = A(x-2) + B(x+2)
\]

Evaluating at \( x = -2 \) gives \( -1 = -4A \), so \( A = 1/4 \).

Evaluating at \( x = 2 \) gives \( -1 = 4B \), so \( B = -1/4 \).

(Alternatively, we could equate the coefficients of the same powers of \( x \) on both sides of the equation:

\[
\begin{align*}
    \text{Coefficient of } x^0: & \quad -1 = -2A + 2B, \\
    \text{Coefficient of } x^1: & \quad 0 = A + B;
\end{align*}
\]

adding two times the second equation to the first equation gives \( -1 = 4B \), so \( B = -1/4 \), and then either equation gives \( A = 1/4 \).

\[
\begin{align*}
    \bar{y} &= \frac{3}{\pi} \int_0^1 \frac{1/4}{x+2} - \frac{1/4}{x-2} \, dx \\
    &= \frac{3}{4\pi} \left[ \ln|x+2| - \ln|x-2| \right]_0^1 \\
    &= \frac{3}{4\pi} \left[ \ln(3 - 2) - (0 - 2) \right] \\
    &= \frac{3 \ln 3}{4\pi}
\end{align*}
\]

4. (a) Find the area of this region.

```
4. Consider the region between the lines \( x = 0 \) and \( x = 1 \),
above the x-axis, and below the curve \( y = \frac{1}{\sqrt{4-x^2}} \).
```

```
\begin{tikzpicture}
    \draw[->] (-0.5,0) -- (1.5,0) node[right] {x};
    \draw[->] (0,-0.5) -- (0,1.5) node[above] {y};
    \draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
    \draw (0.5,0.5) node[above right] {1/2};
\end{tikzpicture}
```
5. (a) Evaluate the limit \( \lim_{x \to 0} \frac{2x - \tan^{-1} x}{\sin^{-1}(3x)} \).

Solution: By L'Hôpital's Rule,
\[
\lim_{x \to 0} \frac{2x - \tan^{-1} x}{\sin^{-1}(3x)} = \lim_{x \to 0} \frac{2 - (1/(1+x^2))}{3/\sqrt{1-(3x)^2}}
= \frac{2 - 1}{3} = \frac{1}{3}
\]

5. (b) Evaluate the limit \( \lim_{x \to 0^+} (\sin x)^{1/(\ln x)} \).

Solution: First, recall that for any base \( b > 0 \) and any real power \( p \),
\[ b^p = e^{p \ln b}. \]
Applying this formula (for small \( x > 0 \)) to \( b = \sin x \) and \( p = 1/(\ln x) \),
\[ (\sin x)^{1/(\ln x)} = e^{(\ln(\sin x))/(\ln x)}. \]
By L'Hôpital’s Rule (applied twice),
\[
\lim_{x \to 0^+} \frac{\ln(\sin x)}{\ln x} = \lim_{x \to 0^+} \frac{(\cos x)/(\sin x)}{1/x} = \lim_{x \to 0^+} \frac{x \cos x}{\sin x} = \lim_{x \to 0^+} \frac{-x \sin x + \cos x}{\cos x} = 0 + 1 = 1.
\]
Thus \( \lim_{x \to 0^+} (\sin x)^{1/(\ln x)} = e^{(\ln(\sin x))/(\ln x)} = e^1 = e. \)

6. (b) Evaluate the improper integral \( \int_1^\infty \frac{2 \ln(x^2 + 1)}{x^3} \, dx \).

Solution:
\[
\int_1^\infty \frac{2 \ln(x^2 + 1)}{x^3} \, dx = \lim_{t \to \infty} \int_1^t \frac{2 \ln(x^2 + 1)}{x^3} \, dx
= \lim_{t \to \infty} \left[ \frac{-\ln(x^2 + 1)}{x^2} + \ln \left( \frac{x^2}{x^2 + 1} \right) \right]_1^t
= \lim_{t \to \infty} \left[ \frac{-\ln(t^2 + 1)}{t^2} + \ln \left( \frac{t^2}{t^2 + 1} \right) \right] - \left( -\ln 2 + \ln \left( \frac{1}{2} \right) \right)
= \lim_{t \to \infty} \left( \frac{-\ln(t^2 + 1)}{t^2} + \ln \left( \frac{t^2}{t^2 + 1} \right) \right) + 2 \ln 2
\]
By L'Hôpital’s Rule,
\[
\lim_{t \to \infty} \left( \frac{-\ln(t^2 + 1)}{t^2} \right) = \lim_{t \to \infty} \left( \frac{-2t/(t^2 + 1)}{2t} \right)
= \lim_{t \to \infty} \left( \frac{-1}{t^2 + 1} \right) = 0,
\]
and \( \lim_{t \to \infty} \left( \frac{t^2}{t^2 + 1} \right) = \lim_{t \to \infty} \left( \frac{2t}{2t} \right) = 1, \) so
\[ \lim_{t \to \infty} \ln \left( \frac{t^2}{t^2 + 1} \right) = \ln 1 = 0. \]
Thus \( \int_1^\infty \frac{2 \ln(x^2 + 1)}{x^3} \, dx = 2 \ln 2. \)
7. Find the solution \( y(x) \) of the initial-value problem

\[
x \frac{dy}{dx} = 2 \ln x, \quad y(1) = 2.
\]

**Solution:** Use separation of variables.

\[
y \frac{dy}{dx} = \frac{2 \ln x}{x} dx
\]

\[
\int y \, dy = \int \frac{2 \ln x}{x} \, dx
\]

\[
\frac{1}{2} y^2 = (\ln x)^2 + C
\]

Use \( y = 2 \) when \( x = 1 \) to solve for \( C \).

\[
\frac{1}{2} \cdot 2^2 = (\ln 1)^2 + C
\]

\[
2 = C
\]

Now solve for \( y \).

\[
\frac{1}{2} y^2 = (\ln x)^2 + 2
\]

\[
y = \pm \sqrt{2(\ln x)^2 + 4}
\]

Since \( y(1) = 2 > 0 \), choose the plus sign.

\[
y = \sqrt{2(\ln x)^2 + 4}
\]

8. A tank contains 100 liters of pure water. Brine that contains 0.2 kg of salt per liter enters the tank at a rate of 10 liters per minute. The solution is kept thoroughly mixed, and the mixed solution drains from the tank at the same rate of 10 liters per minute. How much salt is in the tank after 30 minutes?

**Solution:** Let

\[
t = \text{time (in min)} \text{ after brine starts to enter,}
\]

\[
y(t) = \text{amount of salt (in kg) in tank at time } t.
\]

The incoming flow rate (10 lit/min) is the same as the outgoing flow rate (10 lit/min), so the total volume \( V \) of the solution in the tank remains constant: \( V = 100 \) lit. Since the solution is kept thoroughly mixed, the concentration of the outgoing solution at time \( t \) the same as the average concentration in the whole tank at time \( t \), which is \( y(t)/100 \) kg/lit. Since the units on \( y \) are kg and the units on \( t \) are min, the units on \( \frac{dy}{dt} \) are kg/min, so the units on all (additive) terms in the equation for \( \frac{dy}{dt} \) should be in units of kg/min.

\[
\frac{dy}{dt} = \left[ \text{ rate salt enters tank } \right] - \left[ \text{ rate salt leaves tank } \right] \quad \text{in kg/min}
\]

\[
= \left( \text{ incoming concentration } \right) \cdot \left( \text{ flow rate } \right)
\]

\[
- \left( \text{ outgoing concentration } \right) \cdot \left( \text{ flow rate } \right)
\]

\[
= \left( 0.2 \frac{\text{kg}}{\text{lit}} \right) \cdot \left( \frac{10}{\text{min}} \right) - \left( y(t) \frac{\text{kg}}{\text{lit}} \right) \cdot \left( \frac{10}{\text{min}} \right)
\]

\[
= \left( 2 - \frac{y(t)}{10} \right) \frac{\text{kg}}{\text{min}} = \frac{20 - y(t)}{10} \frac{\text{kg}}{\text{min}}
\]

In addition, since the tank initially contains pure water,

\[
y(0) = 0 \text{ kg.}
\]

So we want to solve the initial value problem:

**Differential Equation:** \[
\frac{dy}{dt} = \frac{20 - y}{10}
\]

**Initial Condition:** \( y(0) = 0 \)

Use separation of variables.

\[
\frac{1}{20 - y} \, dy = \frac{1}{10} \, dt
\]

\[
\int \frac{1}{20 - y} \, dy = \int \frac{1}{10} \, dt
\]

\[
- \ln |20 - y| = \frac{t}{10} + C
\]

Use \( y = 0 \) when \( t = 0 \) to solve for \( C \).

\[
- \ln |20 - 0| = 0 + C
\]

\[
- \ln 20 = C
\]

Now solve for \( y \).

\[
- \ln |20 - y| = \frac{t}{10} - \ln 20
\]

\[
\ln |20 - y| = \ln 20 - \frac{t}{10}
\]

\[
|20 - y| = 20 e^{-t/10}
\]

\[
20 - y = \pm 20 e^{-t/10}
\]

Since \( y(0) = 0 \), choose the plus sign.

\[
20 - y = 20 e^{-t/10}
\]

\[
y(t) = 20 - 20 e^{-t/10}
\]

\[
y(30) = 20 - 20 e^{-30/10} = 20 - 20 e^{-3}
\]

After 30 minutes, there are \( 20 - 20 e^{-3} \) kg of salt in the tank.