1. Compute the derivatives of the following functions. Do not simplify your answers.

   (a) (4 points) $g(t) = \tan^{-1}\left(\frac{5t+3}{t^2+4}\right)$

   $$g'(t) = \frac{1}{1+\left(\frac{5t+3}{t^2+4}\right)^2} \cdot \frac{5\cdot(t^2+4) - 2t\cdot(5t+3)}{(t^2+4)^2}$$

   (b) (4 points) $f(x) = \sqrt{\cos^2 x + 5x^7}$

   $$f'(x) = \frac{-2\cos x \sin x + 35x^6}{2\sqrt{\cos^2 x + 5x^7}}$$

   (c) (4 points) $y = x^{\sqrt{x}}$

   $$\begin{aligned} \ln y &= \ln x^{\sqrt{x}} \\ &= \sqrt{x}\cdot \ln x \\ \frac{1}{y}\cdot y' &= \frac{1}{2\sqrt{x}}\cdot \ln x + \sqrt{x}\cdot\frac{1}{x} \\ y' &= x^{\sqrt{x}}\cdot\left(\frac{1}{2\sqrt{x}}\cdot \ln x + \sqrt{x}\cdot\frac{1}{x}\right) \end{aligned}$$

2. (10 points) Consider an object moving along the parametrized curve with equations:

$$x = \sqrt{t}, \quad y = \frac{t^2}{16}$$

where t is in the time interval $[1,5]$ seconds. Calculate the maximum speed of the object in the time interval.

Let v be the speed. Then $v = \sqrt{\dot{x}^2 + \dot{y}^2}$.

We will maximize $v^2 = \dot{x}^2 + \dot{y}^2$

$\dot{x} = \dfrac{1}{2\sqrt{t}} \qquad \dot{y} = \dfrac{t}{8}$

$v^2 = \dfrac{1}{4t} + \dfrac{t^2}{64}$

$\dfrac{d}{dt}v^2 = \dfrac{-1}{4t^2} + \dfrac{t}{32} = \dfrac{-8+t^3}{32t^2}$

The critical values are $t = 0, 2$. Only $t = 2$ is in the domain.

We must calculate v^2 at $t = 1, 2, 5$.

$v^2(1) = \dfrac{17}{64} = 0.265625$

$v^2(2) = \dfrac{3}{16} = 0.1875$

$v^2(5) = \dfrac{141}{320} = 0.440625$

The maximum speed is $v = \sqrt{\dfrac{141}{320}} \approx 0.6638$

3. (8 points) Find the critical numbers of the function $F(x) = x^{1/3}\left(x^2+2x\right)$

$$\begin{aligned} F'(x) &= \frac{1}{3}x^{-2/3}\left(x^2+2x\right) + x^{1/3}(2x+2) \\ &= \frac{x^2+2x+3x(2x+2)}{3x^{2/3}} \\ &= \frac{7x^2+8x}{3x^{2/3}} \\ &= \frac{7}{3}x^{4/3} + \frac{8}{3}x^{1/3} \\ &= \frac{1}{3}x^{1/3}(7x+8) \end{aligned}$$

Solve $0 = x^{1/3}$ *to get* $x = 0$.

Solve $0 = 7x+8$ *to get* $x = -\frac{8}{7}$

The critical numbers of $F(x)$ *are* $x = 0, -\frac{8}{7}$.

4. (8 points) Check that the point $(2,-1)$ is on the curve $x^3+3xy+2y^3 = 0$. Use the tangent line to approximate the x-coordinate of a point on the curve if the y-coordinate is -1.03.

Check: $(2)^3+3(2)(-1)+2(-1)^3 = 8-6-2 = 0$

Compute the slope of the tangent line at $(2,-1)$*:*

$$\begin{aligned} 0 &= 3x^2+3y+3xy'+6y^2y' \\ &= 3(2)^2+3(-1)+3(2)y'+6(-1)^2y' \\ &= 9+12y' \\ -\frac{3}{4} &= y' \end{aligned}$$

The equation of the tangent line at $(2,-1)$ *is* $y+1 = -\frac{3}{4}(x-2)$

Use the tangent line to approximate x:

$$\begin{aligned} (-1.03)+1 &= -\frac{3}{4}(x-2) \\ 0.04 &= x-2 \\ 2.04 &= x \end{aligned}$$

5. (0 total points) The length of a rectangle increases by 3 feet per minute while the width decreases by 2 feet per minute. When the length is 15 feet and the width is 8 feet, what is the rate at which the following changes. Make sure to state whether the rate is increasing or decreasing and include units.

   (a) (4 points) The area.

   Let x be the length of the rectangle and y be the width.

$$\begin{aligned} A &= xy \\ \frac{dA}{dt} &= \frac{dx}{dt}\,y + x\,\frac{dy}{dt} \\ &= 3\cdot 8 - 15\cdot 2 \\ &= -6 \end{aligned}$$

   The area is decreasing at a rate of 6 ft^2/min.

   (b) (4 points) The perimeter.

$$\begin{aligned} P &= 2x + 2y \\ \frac{dP}{dt} &= 2\frac{dx}{dt} + 2\frac{dy}{dt} \\ &= 2\cdot 3 - 2\cdot 2 \\ &= 2 \end{aligned}$$

   The perimeter is increasing at a rate of 2 ft/min.

   (c) (4 points) The length of the diagonal.

$$\begin{aligned} D &= \sqrt{x^2+y^2} \\ \frac{dD}{dt} &= \frac{2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt}}{2\sqrt{x^2+y^2}} \\ &= \frac{x\,\frac{dx}{dt} + y\,\frac{dy}{dt}}{\sqrt{x^2+y^2}} \\ &= \frac{15\cdot 3 - 8\cdot 2}{\sqrt{15^2+8^2}} \\ &= \frac{29}{17} \\ &\approx 1.706 \end{aligned}$$

   The diagonal is increasing at a rate of $\dfrac{29}{17}$ *ft/min.*