1. Determine if the following limits exist. If they exist, compute them. Justify your answers.

   (a) (4 points) $\lim_{x\to-\infty} \frac{x^2-3x}{\sqrt{6x^4+5}}$

$$\begin{aligned}\lim_{x\to-\infty} \frac{x^2-3x}{\sqrt{6x^4+5}} &= \lim_{x\to-\infty} \frac{(x^2-3x)\cdot\frac{1}{x^2}}{\sqrt{6x^4+5}\cdot\frac{1}{x^2}} \\ &= \lim_{x\to-\infty} \frac{1-\frac{3}{x}}{\sqrt{6+\frac{5}{x^4}}} \\ &= \frac{1}{\sqrt{6}}\end{aligned}$$

   (b) (4 points) $\lim_{h\to0} \left(\frac{3}{h}-\frac{6}{2h-h^2}\right)$

$$\begin{aligned}\lim_{h\to0} \left(\frac{3}{h}-\frac{6}{2h-h^2}\right) &= \lim_{h\to0} \frac{3(2-h)-6}{2h-h^2} \\ &= \lim_{h\to0} \frac{-3h}{2h-h^2} \\ &= \lim_{h\to0} \frac{-3}{2-h} \\ &= -\frac{3}{2}\end{aligned}$$

   (c) (4 points) $\lim_{x\to3} \frac{2x^2-4x+7}{x-3}$

The numerator is going to 13 *and the denominator is going to* 0.

$x-3$ *is negative when* $x<3$ *and positive when* $x>3$.

Thus $\lim_{x\to3^+} \frac{2x^2-4x+7}{x-3} = \infty$ *and* $\lim_{x\to3^-} \frac{2x^2-4x+7}{x-3} = -\infty$

Since the left and right limits are not equal, the limit does not exist.

2. (7 points) Do not use any differentiation formulas in this problem. Use limits where appropriate. Find the slope of the tangent line to the curve $y = x^2 + 7x$ at the point $(1, 8)$.

   Let m be the slope of the tangent line at $x = 1$. Then

$$\begin{aligned} m &= \lim_{h\to 0} \frac{y(1+h) - y(1)}{h} \\ &= \lim_{h\to 0} \frac{(1+h)^2 + 7(1+h) - 8}{h} \\ &= \lim_{h\to 0} \frac{1 + 2h + h^2 + 7 + 7h - 8}{h} \\ &= \lim_{h\to 0} \frac{9h + h^2}{h} \\ &= \lim_{h\to 0} 9 + h \\ &= 9 \end{aligned}$$

3. (7 points) A particle is travelling in a straight line. Its position is given by $x = \left(t^2 + t\right) e^t$, where x is in feet and t is in seconds. Find all times when the acceleration of the particle is zero.

   We must find when the 2nd derivative is zero.

$$\begin{aligned} \frac{dx}{dt} &= \left(t^2 + 3t + 1\right) e^t \\ \frac{d^2x}{dt^2} &= \left(t^2 + 5t + 4\right) e^t \\ 0 &= \left(t^2 + 5t + 4\right) e^t \\ 0 &= t^2 + 5t + 4 \qquad \textit{because } e^t > 0 \\ &= (t+1)(t+4) \end{aligned}$$

   Thus the two times are $t = -1$ *and* $t = -4$.

4. (7 points) Let c be a constant. Define $F(x)$ by the piecewise formula

$$F(x) = \begin{cases} cx^2 - 7x & \text{if } x \leq -2; \\ cx + 8 & \text{if } x > -2. \end{cases}$$

Find a value of c that makes F continuous on $(-\infty, \infty)$. Justify your answer. (Your justification should involve limits).

First note that, for any c, $F(x)$ is continuous if $x \neq -2$. Indeed, if $x > -2$ then $F(x)$ is a linear function and if $x < -2$ then $F(x)$ is a quadratic polynomial. These functions are always continuous.

Now consider $x = -2$. We must show that $\lim_{x \to -2} F(x)$ *exists, and equals* $F(-2)$.

Note that $F(-2) = 4c + 14$

To show that the limit exists, we must show that the left and right limits are equal.

$$\begin{aligned} \lim_{x \to -2^+} F(x) &= \lim_{x \to -2^+} cx + 8 \\ &= 8 - 2c \\ \lim_{x \to -2^-} F(x) &= \lim_{x \to -2^-} cx^2 - 7x \\ &= 4c + 14 \end{aligned}$$

Note that the left limit equals $F(-2)$.

Thus it is enough to find a value c so that $8 - 2c = 4c + 14$.

$c = -1$ *is the value we want.*

5. (7 points) Calculate the equation of the tangent line to $g(x) = \left|x^2 - 5x\right|$ at $x = 4$.

$$g(x) = \begin{cases} -x^2 + 5x & \text{if } 0 < x < 5; \\ x^2 - 5x & \text{otherwise.} \end{cases}$$

$$g'(x) = \begin{cases} -2x + 5 & \text{if } 0 < x < 5; \\ 2x - 5 & \text{otherwise.} \end{cases}$$

$g'(4) = -3, \qquad g(4) = 4, \qquad y - 4 = -3(x - 4)$

6. (10 points) Find the equations of **all** tangent lines to the curve $y = \frac{3x+5}{2x+7}$ that are parallel to the line $11x - 25y = 2$.

Note that the slope of the given line is $m = \frac{11}{25}$.

We first find all x values where $\frac{dy}{dx} = \frac{11}{25}$.

$$\begin{aligned}
\frac{11}{25} &= \frac{dy}{dx} \\
&= \frac{3(2x+7) - 2(3x+5)}{(2x+7)^2} \\
&= \frac{11}{(2x+7)^2} \\
\frac{1}{25} &= \frac{1}{(2x+7)^2} \\
(2x+7)^2 &= 25 \\
2x+7 &= \pm 5
\end{aligned}$$

Thus $x = -6, -1$

Then $y(-6) = \frac{13}{5}$ *and* $y(-1) = \frac{2}{5}$

The lines are $y - \frac{13}{5} = \frac{11}{25}(x+6)$ *and* $y - \frac{2}{5} = \frac{11}{25}(x+1)$