1. Determine if the following limits exist. If they exist, compute them. Justify your answers.

   (a) (4 points) $\lim_{x\to 2} \frac{x^2-4}{2x^2-3x-2}$

   This is a $\frac{0}{0}$ limit. We remove the singularity.

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-4}{2x^2-3x-2} &= \lim_{x\to 2} \frac{(x-2)(x+2)}{(2x+1)(x-2)} \\ &= \lim_{x\to 2} \frac{x+2}{2x+1} \\ &= \frac{4}{5}\end{aligned}$$

   (b) (4 points) $\lim_{t\to 0} \left(\frac{3}{5t^2+3t} - \frac{1}{t}\right)$

   Combine over a common denominator.

$$\begin{aligned}\lim_{t\to 0} \left(\frac{3}{5t^2+3t} - \frac{1}{t}\right) &= \lim_{t\to 0} \left(\frac{3}{5t^2+3t} - \frac{1}{t}\cdot\frac{5t+3}{5t+3}\right) \\ &= \lim_{t\to 0} \frac{-5t}{5t^2+3t} \\ &= \lim_{t\to 0} \frac{-5}{5t+3} \\ &= -\frac{5}{3}\end{aligned}$$

   (c) (4 points) $\lim_{x\to 2} \frac{\sqrt{3x-2}}{2x^2-7x+6}$

   This is a $\frac{2}{0}$ limit. It is an infinite limit and we must use one-sided limits.
   Factor the denominator to get $2x^2-7x+6 = (x-2)(2x-3)$. Conclude that $2x^2-7x+6$ is positive when $x > 2$ and negative when $\frac{3}{2} < x < 2$.
   Near $x = 2$ the numerator is near 2 and hence is positive.
   Thus $\lim_{x\to 2^+} \frac{\sqrt{3x-2}}{2x^2-7x+6} = \infty$ because the top is positive and the bottom is positive.
   Also $\lim_{x\to 2^-} \frac{\sqrt{3x-2}}{2x^2-7x+6} = -\infty$ because the top is positive and the bottom is negative.
   We conclude $\lim_{x\to 2} \frac{\sqrt{3x-2}}{2x^2-7x+6}$ does not exist.

2. (7 points) Use the limit definition of the derivative on this problem. Find the slope of the tangent line to the curve $y = \sqrt{3x+7}$ at the point $(3,4)$.

   Compute the derivative at $x = 3$.

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{3x+7} - \sqrt{3\cdot 3+7}}{x-3} &= \lim_{x\to 3} \frac{\sqrt{3x+7}-4}{x-3} \cdot \frac{\sqrt{3x+7}+4}{\sqrt{3x+7}+4} \\
&= \lim_{x\to 3} \frac{3x+7-16}{(x-3)\left(\sqrt{3x+7}+4\right)} \\
&= \lim_{x\to 3} \frac{3}{\sqrt{3x+7}+4} \\
&= \frac{3}{8}
\end{aligned}$$

   The slope of the tangent line is $m_{\tan} = \frac{3}{8}$.

3. (7 points) Calculate the equation of the tangent line to $f(t) = 7t^2 \cdot \sin(t)$ at $t = \pi$.

   The tangent line is $y - b = m(t-a)$ *where* $a = \pi$, $b = f(\pi) = 0$ *and* $m = f'(\pi)$.

   $f'(t) = 14t \cdot \sin(t) + 7t^2 \cdot \cos(t)$

   $f'(0) = -7\pi^2$

   The equation of the tangent line is $y = -7\pi^2(t-\pi)$

4. (8 points) Let c be a constant. Define $F(x)$ by the piecewise formula

$$F(x) = \begin{cases} cx^2 - 3x & \text{if } x \le -1; \\ cx + 11 & \text{if } x > -1. \end{cases}$$

Find a value of c that makes F continuous on $(-\infty, \infty)$. Justify your answer. (Your justification should involve limits).

For any value c, $F(x)$ is quadratic if $x < -1$ and linear if $x > 1$. Thus the only value where is might not be continuous is $x = -1$.

We must find c so that $\lim\limits_{x \to -1} F(x) = F(-1)$. Note that $F(-1) = c + 3$.

For the limit to exist, we must have $\lim\limits_{x \to -1^-} F(x) = \lim\limits_{x \to -1^+} F(x)$.

$$\begin{aligned} \lim_{x \to -1^-} F(x) &= \lim_{x \to -1^-} cx^2 - 3x \\ &= c + 3 \\ \lim_{x \to -1^+} F(x) &= \lim_{x \to -1^+} cx + 11 \\ &= 11 - c \end{aligned}$$

Thus we must have $c + 3 = 11 - c$, or $c = 4$.

This makes F continuous everywhere because $\lim_{x \to -1^-} F(x) = F(-1)$.

5. (8 points) Find **two** different points on the curve $y = x^3$ at which the tangent line passes through the point $(6, 0)$.

Let (a, b) be a point on the curve with the given property.

The tangent line looks like $y - b = m(x - a)$. We have $b = a^3$ and $x = 6$, $y = 0$.

$m = \left.\dfrac{dy}{dx}\right|_{x=a} = 3a^2$

$$\begin{aligned} 0 - a^3 &= 3a^2(6 - a) \\ 2a^3 - 18a^2 &= 0 \\ 2a^2(a - 9) &= 0 \\ a &= 0, 9 \end{aligned}$$

The two points are $(0, 0)$ and $(9, 729)$.

6. (8 points) Find the equations of **all** tangent lines to the curve $y = \dfrac{4x-5}{2x-3}$ that are parallel to the line $2x+9y=7$.

First compute $\dfrac{dy}{dx} = -\dfrac{2}{(2x-3)^2}$

The slope of $2x+9y=7$ *is* $m=-\dfrac{2}{9}$.

Solve $-\dfrac{2}{(2x-3)^2} = -\dfrac{2}{9}$ *to get* $x = 0, 3$.

The points on the curve are $(0, 5/3)$ *and* $(3, 7/3)$.

The lines are $y - \dfrac{5}{3} = -\dfrac{2}{9}x$ *and* $y - \dfrac{7}{3} = -\dfrac{2}{9}(x-3)$.