$\boxed{1}$ (12 points) Determine if the following limits exist. If they exist, compute them. Justify your answers.

(a) (4 points) $\lim_{x\to\infty}\left(\sqrt{4x^2-3x}-2x\right)$

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{4x^2-3x}-2x\right) &= \lim_{x\to\infty}\left(\sqrt{4x^2-3x}-2x\right)\cdot\frac{\sqrt{4x^2-3x}+2x}{\sqrt{4x^2-3x}+2x}\\ &= \lim_{x\to\infty}\frac{4x^2-3x-4x^2}{\sqrt{4x^2-3x}+2x}\\ &= \lim_{x\to\infty}\frac{-3x}{\sqrt{4x^2-3x}+2x}\cdot\frac{1/x}{1/x}\\ &= \lim_{x\to\infty}\frac{-3}{\sqrt{4-3/x}+2}\\ &= -\frac{3}{4}\end{aligned}$$

(b) (4 points) $\lim_{x\to 3}\frac{\sqrt{2x-1}}{x^2-6x+8}$

$f(x)=\frac{\sqrt{2x-1}}{x^2-6x+8}$ *is continuous at* $x=3$.

$\lim_{x\to 3} f(x) = f(3) = -\sqrt{5}$

(c) (4 points) $\lim_{x\to 2}\frac{2x^2-x-6}{x^3-5x^2+12}$

*This is a* $\frac{0}{0}$ *limit.*

$2x^2-x-6=(x-2)(2x+3)$

*Use long division to see that* $x^3-5x^2+12=(x-2)(x^2-3x-6)$

$$\begin{aligned}\lim_{x\to 2}\frac{2x^2-x-6}{x^3-5x^2+12} &= \lim_{x\to 2}\frac{2x+3}{x^2-3x-6}\\ &= -\frac{7}{8}\end{aligned}$$

$\boxed{2}$ (7 points) Do not use any differentiation formulas in this problem. Use limits where appropriate. Calculate the derivative of the function $f(x) = \dfrac{1}{3-x}$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\cdot\left(\frac{1}{3-x-h} - \frac{1}{3-x}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\cdot\frac{(3-x)-(3-x-h)}{(3-x)(3-x-h)} \\
&= \lim_{h\to 0} \frac{1}{h}\cdot\frac{h}{(3-x)(3-x-h)} \\
&= \lim_{h\to 0} \frac{1}{(3-x)(3-x-h)} \\
&= \frac{1}{(3-x)^2}
\end{aligned}
$$

$\boxed{3}$ (7 points) Calculate the equation of the tangent line to $g(x) = \dfrac{1+x}{1+x+x^2}$ at $x = 2$.

$$g(2) = \frac{3}{7}$$

$$g'(x) = \frac{1\cdot(1+x+x^2) - (1+x)(1+2x)}{(1+x+x^2)^2}$$

$$g'(2) = \frac{7-3\cdot 5}{7^2} = -\frac{8}{49}$$

*The tangent line is* $y - \dfrac{3}{7} = -\dfrac{8}{49}(x-2)$

$\boxed{4}$ (7 points) Let $H(x) = \begin{cases} (x-1)^2 & \text{if } x < 0; \\ e^{x^2} & \text{if } x \geq 0. \end{cases}$

Is $H(x)$ a continuous function? Use limits to give a careful justification of your answer.

*$H(x) = (x-1)^2$ if $x < 0$. This is continuous.*

*$H(x) = e^{x^2}$ if $x > 0$. This is continuous.*

*Thus $H(x)$ is continuous if $x \neq 0$*

$H(0) = e^{0^2} = 1$

$$\lim_{x \to 0^+} H(x) = \lim_{x \to 0^+} e^{x^2} = 1$$

$$\lim_{x \to 0^-} H(x) = \lim_{x \to 0^-} (x-1)^2 = 1$$

*Thus $\lim_{x \to 0} H(x) = H(0)$ and $H(x)$ is continuous everywhere.*

$\boxed{5}$ (7 points) Find all the points $(a, b)$ on the curve $y = \left(x^2 - 15\right) e^x$ where the tangent line is horizontal.

$$\begin{aligned} \frac{dy}{dx} &= \left(x^2 + 2x - 15\right) e^x \\ \left(x^2 + 2x - 15\right) e^x &= 0 \\ \left(x^2 + 2x - 15\right) &= 0 \qquad \text{since } e^x > 0 \\ (x-3)(x+5) &= 0 \\ x &= -5, 3 \end{aligned}$$

*The points are $\left(-5, 10e^{-5}\right)$ and $\left(3, -6e^3\right)$.*

*The solution can also be written $(-5, 0.067)$ and $(3, -120.5)$.*

$\boxed{6}$ (10 points) Find all points $(a, b)$ on the curve $y = x^3 - 4x^2$ where the tangent line has a $x$-intercept of 6.

*Let $x = a$ be the $x$-coordinate of such a point.*

*The tangent line has the form $y - b = m(x - a)$*

$b = a^3 - 4a^2$

$\frac{dy}{dx} = 3x^2 - 8x$

$m = 3a^2 - 8a$

*The general tangent line is $y - a^3 + 4a^2 = (3a^2 - 8a)(x - a)$*

*We plug in $x = 6$ and $y = 0$ to get*

$$\begin{aligned}
-a^3 + 4a^2 &= (3a^2 - 8a)(6 - a) \\
-a^3 + 4a^2 &= -3\,a^3 + 26\,a^2 - 48\,a \\
0 &= -2a^3 + 22a^2 - 48a \\
&= -2a\left(a^2 - 11a + 24\right) \\
&= -2a(a - 3)(a - 8)
\end{aligned}$$

*Thus $a = 0$, 3 or 8.*

*The points are $(0, 0)$, $(3, -9)$ and $(8, 256)$.*