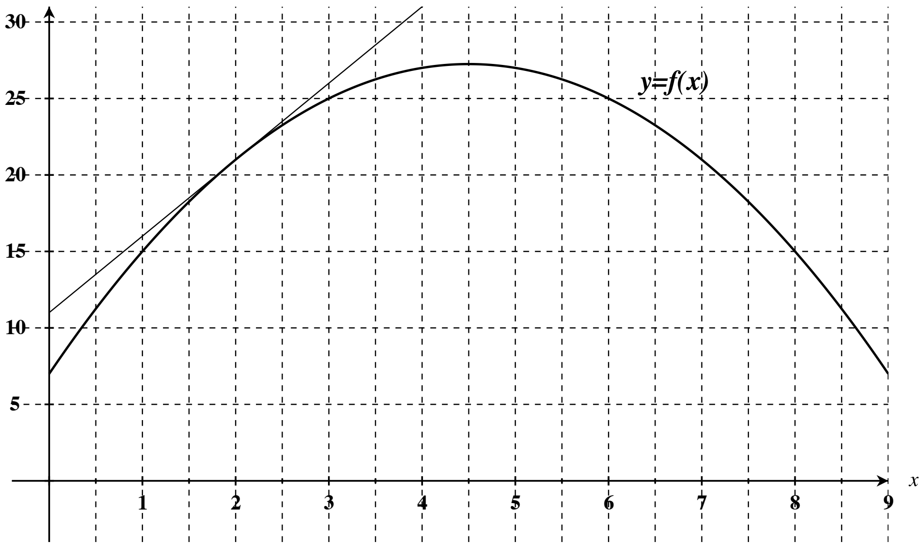
Math 112

Solutions for Group Activity: Tangents and Secants

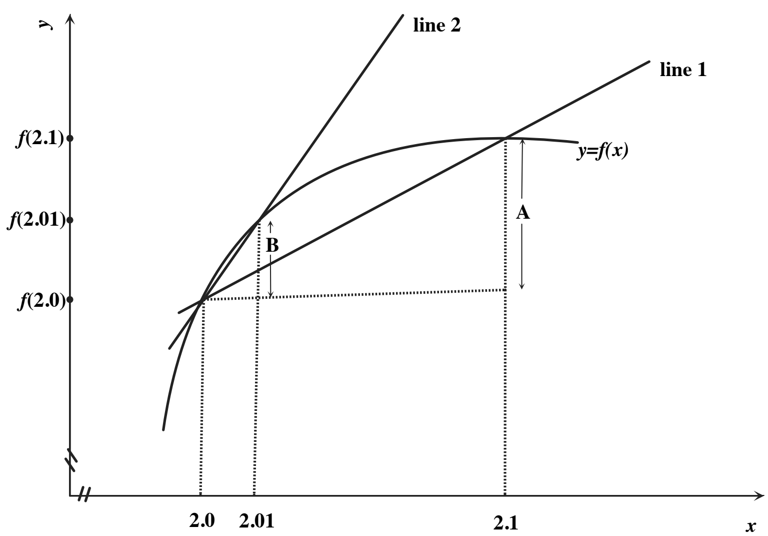
The graph below is of a function $y = f(x)$.



1. Draw a tangent line to the graph of $f(x)$ at $x = 2$ and compute its slope. What is the functional notation for the slope you just computed?

   ANSWER: The line is drawn above. I used the points $(3.75, 30)$ and $(0.75, 15)$ to compute the slope. Using functional notation, we'd write $\boxed{f'(2) = 5}$. (This is an approximation from the graph. Your answer should be close to 5 but may not be exactly 5.)

2. Below is a magnification of the graph of $f(x)$ near $x = 2$. (It's not to scale and I've exaggerated the curviness of $f(x)$.) There are three points marked on the y-axis. Their heights are $f(2.0)$, $f(2.1)$ and $f(2.01)$. Label them correctly.



3. In terms of $f(2.0)$, $f(2.1)$ and $f(2.01)$, what are the values of the lengths marked A and B and the slopes of line 1 and line 2 in the picture above?

$A = f(2.1) - f(2.0)$

$B = f(2.01) - f(2.0)$

$$\text{slope of line 1} = \frac{f(2.1) - f(2.0)}{0.1}$$

$$\text{slope of line 2} = \frac{f(2.01) - f(2.0)}{0.01}$$

4. Which is closer to the slope of the tangent line to $f(x)$ at $x = 2$: the slope of line 1 or the slope of line 2?

   ANSWER: $\boxed{\text{the slope of line 2}}$

5. How might you draw a line whose slope is even closer to the slope of the tangent to $f(x)$ at $x = 2$?

   ANSWER: You could draw the secant line through the graph of $f(x)$ at $x = 2.0$ and $x = 2.001$, for example. Other answers are acceptable.

6. The formula for $f(x)$ is: $f(x) = -x^2 + 9x + 7$.

   (a) The slope of line 1 is given by $\dfrac{f(2.1) - f(2.0)}{0.1}$. Use the formula for $f(x)$ to compute the exact value of this slope.

   ANSWER: $\boxed{4.9}$

   (b) Use the formula for $f(x)$ to compute the exact value of the slope of line 2.

   ANSWER: $\boxed{4.99}$

   (c) Your work in parts (a) and (b) forms the first two lines of the following table. We've completed the next two lines for you. Use the pattern established in the table to complete the last two lines of the table. (Do not try to use your calculator to finish the table—just follow the pattern that you see emerging. Most calculators will round and the rounding error will cause the pattern stop.)

h	$2+h$	$f(2+h)$	f(2)	$f(2+h) - f(2)$	$\dfrac{f(2+h)-f(2)}{h}$
0.1	2.1	21.49	21	0.49	4.9
0.01	2.01	21.0499	21	0.0499	4.99
0.001	2.001	21.004999	21	0.004999	4.999
0.0001	2.0001	21.00049999	21	0.00049999	4.9999
0.00001	2.00001	21.0000499999	21	0.0000499999	4.99999
0.000001	2.000001	21.000004999999	21	0.000004999999	4.999999

   (d) Do you agree that the right-most column of the table gives slopes of secant lines that are getting closer and closer to the tangent line to $f(x)$ at $x = 2$? What would you predict is the exact value of $f'(2)$, the slope of the tangent line to $f(x)$ at $x = 2$? Is this close to the slope you computed using the graph in #1?

   ANSWER: Yes, the table gives slopes of secant lines that get progressively closer to the tangent line to $f(x)$ at $x = 2$. I would predict that the exact value of this slope is 5 and this is what I got in #1.

7. We'll use a similar method to compute $f'(3)$, the slope of the line tangent to $f(x)$ at $x = 3$, but this time we'll use algebra to make a smaller table. Again, the formula for $f(x)$ is:

$$f(x) = -x^2 + 9x + 7.$$

   (a) Compute and simplify the expression $\dfrac{f(3+h) - f(3)}{h}$.

   SOLUTION:

$$\begin{aligned}
f(3+h) &= -(3+h)^2 + 9(3+h) + 7 \\
&= -(9 + 6h + h^2) + 9(3+h) + 7 \\
&= -9 - 6h - h^2 + 27 + 9h + 7 \\
&= -h^2 + 3h + 25. \\
f(3) &= -9 + 27 + 7 = 25. \\
\frac{f(3+h) - f(3)}{h} &= \frac{-h^2 + 3h + 25 - 25}{h} = \frac{-h^2 + 3h}{h} = -h + 3
\end{aligned}$$

   ANSWER: $\boxed{-h+3}$

   (b) What is the graphical interpretation of the expression in part (a)? (Your answer should begin "It is the slope of the...".)

   ANSWER: It is the slope of the secant line through the graph of $f(x)$ at $x = 3$ and $x = 3 + h$.

(c) Use your answer to part (a) to quickly fill in the following table:

h	$\dfrac{f(3+h)-f(3)}{h}$
0.1	2.9
0.01	2.99
0.001	2.999
0.0001	2.9999

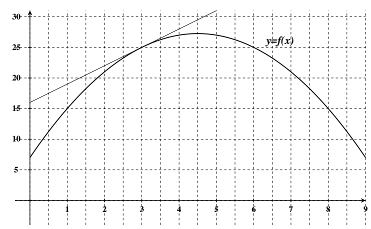
<— This is just $3-h$.

(d) Do you agree that the right-most column of the table gives slopes of secant lines that are getting closer and closer to the tangent line to $f(x)$ at $x=3$? What would you predict is the exact value of $f'(3)$, the slope of the tangent line to $f(x)$ at $x=3$?

ANSWER: Yes, the table gives slopes of secant lines that get progressively closer to the tangent line to $f(x)$ at $x=3$. I'd predict that $f'(3)=3$.

(e) On the graph at the beginning of this activity, draw the line tangent to $f(x)$ at $x=3$ and compute its slope. Is it close to the value you just found for $f'(3)$?

ANSWER: Draw this line and compute its slope. The slope should be close to 3:



8. Finally, let's find a function that gives $f'(a)$, the slope of the tangent line to $f(x)$ at $x=a$. Again,

$$f(x)=-x^2+9x+7.$$

(a) Compute and simplify the expression $\dfrac{f(a+h)-f(a)}{h}$.

SOLUTION:

$$\begin{aligned} f(a+h) &= -(a+h)^2+9(a+h)+7 \\ &= -a^2-2ah-h^2+9a+9h+7. \\ f(a) &= -a^2+9a+7 \\ \frac{f(a+h)-f(a)}{h} &= \frac{-2ah-h^2+9h}{h} = -2a-h+9. \end{aligned}$$

ANSWER: $\boxed{-2a-h+9}$

(b) What is the graphical interpretation of the expression in part (a)? (Your answer should begin "It is the slope of the...".)

ANSWER: It is the slope of the secant line through the graph of $f(x)$ at $x=a$ and $x=a+h$.

(c) If you take progressively smaller values for h, your answer to part (a) approaches some value that depends on a. What is that value, what is its graphical interpretation, and how would you use functional notation to express it?

ANSWER: As h gets smaller and smaller, the slope we just found gets closer and closer to $-2a+9$. This gives the slope of the tangent line to $f(x)$ at $x=a$. We'd denote it $f'(a)$. That is, the slope of the tangent line to $f(x)$ at $x=a$ is given by $\boxed{f'(a)=-2a+9.}$

(d) Use your answer to part (c) to compute $f'(2)$ and $f'(3)$. Do these match your previous computations?

ANSWER: According to the formula $f'(a)=-2a+9$:

- $f'(2)=-2(2)+9=5$, which is what we got before
- $f'(3)=-2(3)+9=3$, which is what we got before

9. Use your answer to part (c) to compute $f'(1)$ and $f'(5)$.

ANSWER: $f'(1)=-2(1)+9=7$ and $f'(5)=-2(5)+9=-1$