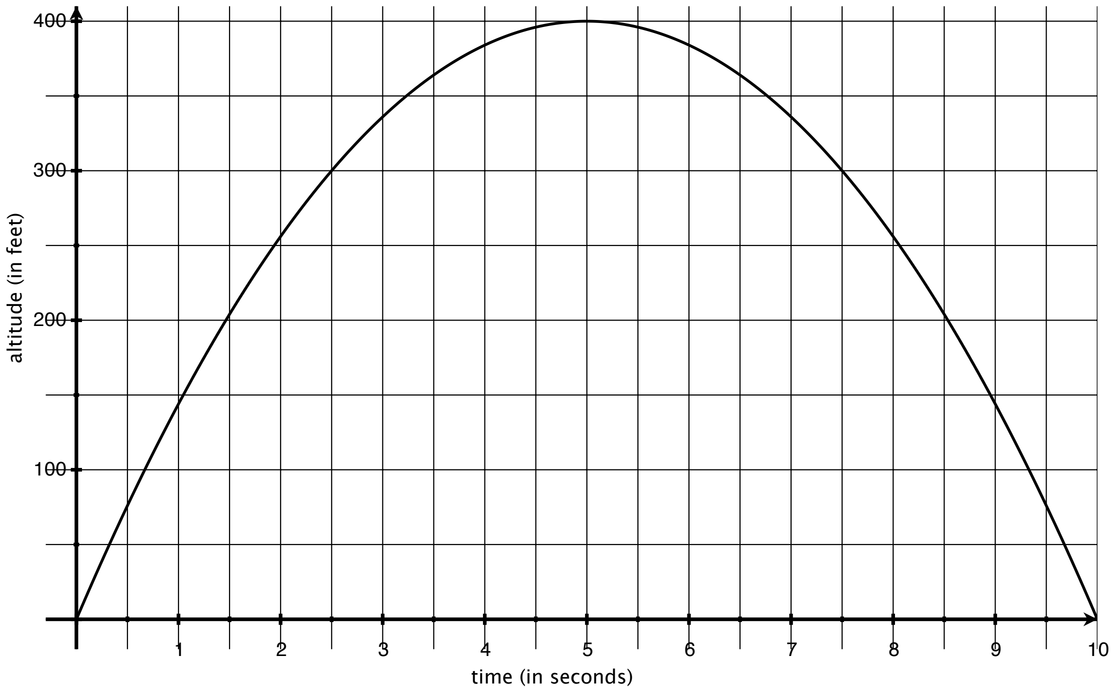
Math 112

Solutions for Group Activity: The Vertical Speed of a Shell

A shell is fired straight up by a mortar. The graph below shows its altitude as a function of time.



The function describing the shell's altitude at time t will be denoted by $f(t)$, where altitude is in feet and t is in seconds.

1. Use the altitude graph to approximate the following quantities:

   ANSWERS: Your answers to #1 should be close to the exact answers in #2.

   (a) the time required for the shell to reach the altitude 300 ft for the first time
   SOLUTION: Find the first time when the height of the graph is 300.

   (b) the change in altitude from time $t = 2$ to $t = 4$
   SOLUTION: Find the difference in the height of the graph from $t = 2$ to $t = 4$.

   (c) the average speed of the shell from time $t = 1$ to time $t = 4$
   SOLUTION: Compute the change in the altitude from $t = 1$ to $t = 4$ and divide by time elapsed.

The formula for the altitude (in feet) after t seconds is given by: $f(t) = 160t - 16t^2$.

2. Use this formula to compute the exact values of the quantities from question #1. Check these values against the approximations you read from the graph.

   (a) the time required for the shell to reach the altitude 300 ft for the first time
   SOLUTION: Use the quadratic formula to solve the equation $300 = 160t - 16t^2$.
   ANSWER: $\boxed{t = 2.5 \text{ seconds}}$

   (b) the change in altitude from time $t = 2$ to $t = 4$
   SOLUTION: Compute $f(4) - f(2)$.
   ANSWER: $f(4) - f(2) = \boxed{128 \text{ feet}}$

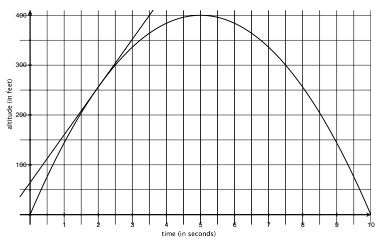
   (c) the average speed of the shell from time $t = 1$ to time $t = 4$
   SOLUTION: Compute the change in altitude ($f(4) - f(1)$ feet) and divide by time elapsed (3 seconds).
   ANSWER: average speed $= \dfrac{f(4) - f(1)}{4 - 1} = \boxed{80 \text{ feet per second}}$

Imagine that there is a tiny speedometer in the shell that records the shell's velocity just as the speedometer in your car records the car's velocity. This speedometer measures the shell's *instantaneous speed*. This is the speed the shell is traveling, not over an interval of time, but at one specific instant. We usually think of speed as distance traveled divided by time elapsed, but over a single instant, distance traveled and time elapsed are both 0 and speed is not so simple. Finding instantaneous speeds is what calculus is all about. We'll start by making some approximations.

3. We'll approximate the shell's instantaneous speed at $t = 2$ by finding the shell's average speed over a tiny time interval (an interval of length 0.01 seconds). We'll do this in two different ways:

   (a) **Graphically:** On the graph of altitude, draw the line whose slope gives the shell's average speed from $t = 2$ to $t = 2.01$. That is, draw the line that goes through the graph of altitude at $t = 2$ and $t = 2.01$. Note that, on this scale, it's difficult to distinguish between $t = 2$ and $t = 2.01$. Do your best to draw such a line, extending it out as far as you can in both directions. Then estimate the coordinates of two points on the line and compute its slope. (Regardless of your choice of points, this gives the desired average speed. Make sure you understand why.)

   SOLUTION: Draw a line that looks like this and compute its slope:



   (b) **Algebraically:** Use the formula for altitude to compute the shell's average speed from $t = 2$ to $t = 2.01$. That is, compute

$$\frac{f(2.01) - f(2)}{0.01}.$$

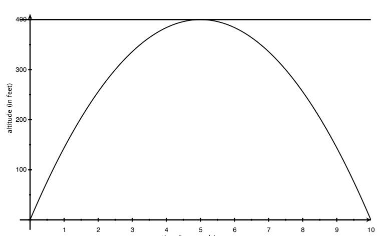
   Compare with your answer from part (a). Would you agree that this is a good approximation of the shell's speed at the instant $t = 2$?

   ANSWER: $\dfrac{f(2.01) - f(2)}{0.01} = \dfrac{256.9584 - 256}{0.01} = \boxed{\text{95.84 feet per second}}$

4. Use the same two methods to approximate the shell's instantaneous speed at $t = 5$, when it reaches its peak height. Instantaneous speed at $t = 5$ will be approximately equal to the average speed from $t = 5$ to $t = 5.01$.

   (a) **Graphically:**

   SOLUTION: Draw a line that looks like this and compute its slope:



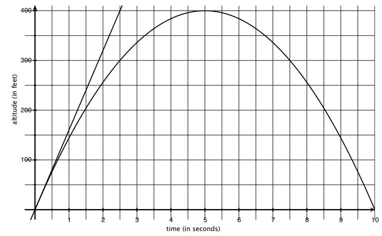
   ANSWER: This is one where the graphical method is easier and gives a more accurate answer than the algebraic method. This line is essentially horizontal and the approximate speed of the shell at $t = 5$ is $\boxed{\text{0 feet per second.}}$ (The shell stops for an instant when it changes its direction of motion from up to down.)

   (b) **Algebraically:**

   ANSWER: $\dfrac{f(5.01) - f(5)}{0.01} = \dfrac{399.9984 - 400}{0.01} = \boxed{\text{-0.16 feet per second}}$

5. Use the same two methods to approximate the shell's instantaneous speed at $t = 0$, the instant it was launched. Instantaneous speed at $t = 0$ will be approximately equal to the average speed from $t = 0$ to $t = 0.01$.

   (a) **Graphically:**
   SOLUTION: Draw a line that looks like this and compute its slope:



   (b) **Algebraically:**
   ANSWER: $\dfrac{f(0.01) - f(0)}{0.01} = \dfrac{1.5984 - 0}{0.01} = \boxed{\text{159.84 feet per second}}$

6. In #3, you approximated the shell's speed at $t = 2$ by computing the average speed over the interval from $t = 2$ to $t = 2.01$. This is a decent approximation of the speed at the instant $t = 2$ but we can get even better approximations by using even shorter time intervals.

   (a) Use the algebraic method to compute the average speed of the shell over the interval from $t = 2$ to $t = 2.001$.
   ANSWER: $\dfrac{f(2.001) - f(2)}{0.001} = \dfrac{256.095984 - 256}{0.001} = \boxed{\text{95.984 feet per second}}$

   (b) Use the algebraic method to compute the average speed of the shell over the interval from $t = 2$ to $t = 2.0001$.
   ANSWER: $\dfrac{f(2.0001) - f(2)}{0.0001} = \dfrac{256.00959984 - 256}{0.0001} = \boxed{\text{95.9984 feet per second}}$

   (c) What would you guess is the exact value of the instantaneous speed of the shell at $t = 2$?
   ANSWER: $\boxed{\text{96 feet per second}}$

   (d) Describe how you could interpret the instantaneous speed of the shell at $t = 2$ on the graph of altitude as the slope of a line.
   ANSWER: $\boxed{\text{It's the slope of the line tangent to } f(t) \text{ at } t = 2.}$