A USER'S GUIDE TO THE BOCKSTEIN SPECTRAL SEQUENCE

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Abstract. We consider a non-traditional viewpoint on the Bockstein spectral sequence.

1. Introduction

The classical Bockstein spectral sequence arises from the short exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/p \to 0$: given a chain complex C of free abelian groups, tensoring it with this short exact sequence gives a short exact sequence of chain complexes,

$$0 \to C \xrightarrow{p} C \to C \otimes \mathbf{Z}/p \to 0.$$

Taking homology produces a long exact sequence in which two of every three terms is the same, which thus can be viewed as an exact couple. The *Bockstein spectral sequence* is the resulting spectral sequence; traditionally (as in [Bro61], [McC01] or [MT68], for example), it is described as follows: the E_1-term is $H_*(C \otimes \mathbf{Z}/p)$, and under some hypotheses, it converges to $(H_*(C)/\text{torsion}) \otimes \mathbf{Z}/p$.

There is an issue: while this presentation is pleasant, it can be useful to view the Bockstein spectral sequence in a way that is more consistent with other spectral sequences. In this traditional view, the E_1-term is singly graded, and the abutment is a bit artificial. It is not hard to form a Bockstein spectral sequence in which the E_1-term is doubly graded, as usual for spectral sequences, and the abutment is essentially the homology of the original chain complex. This note has three goals, the first of which is to give such a presentation.

Another issue is convergence. As is well understood in the classical situation, copies of $\mathbf{Q}$ in the target are invisible, and the groups $\mathbf{Z}_{(p)}$ and $\mathbf{Q}/\mathbf{Z}_{(p)}$ are indistinguishable. A common way to avoid these problems is to work with chain complexes whose homology groups are finitely generated (as McCleary does, for instance). The second goal of this paper is to give some simple convergence criteria.

The basic set-up of the spectral sequence can be generalized; indeed, it is not uncommon in the literature (see [MRW77], for example) to see spectral sequences arising as follows: let C be a chain complex, and consider a chain map $\theta : C \to C$. Assume, for instance, that θ is injective. Then there is a short exact sequence of chain complexes of the form

$$0 \to C \xrightarrow{\theta} C \to \operatorname{coker} \theta \to 0.$$

Taking homology yields a long exact sequence in which two of every three terms is the same, and this gives an exact couple. It is reasonable to call the resulting spectral sequence a "Bockstein spectral sequence"; it starts with the homology of

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coker θ, and abuts to the homology of C. Unfortunately, most reference books don't seem to discuss this situation, or if they do, they don't treat it as a Bockstein spectral sequence. The third goal of this note is to provide a treatment of this more general situation.

The main results are described in Theorem 3.8 and 3.10; we (briefly) consider a few examples at the end of the paper.

The experts certainly know everything in this paper. The non-experts may not, though, and they are the intended audience.

Acknowledgments: I learned about this presentation of the Bockstein spectral sequence from Hal Sadofsky. I was motivated to write this note by a conversation about this at Mui's 60th birthday conference in Hanoi, 2004; if I recall correctly, some of the people involved were Tilman Bauer, Mike Hill, and Mark Behrens.

2. Conventions

We use upper indices for graded objects, so we write $C = \bigoplus C^t$ for a chain complex, and we write H^*C for its homology. In chain complexes, the degree $|d|$ of the differential will be unspecified; usually in applications it will be 1 or -1.

If $C = \bigoplus_n C^n$ is a graded abelian group, then its *suspension* ΣC is defined by $(\Sigma C)^n = C^{n-1}$. Note that if C is a chain complex, then $H^k(\Sigma^n C) = H^{k-n}C$.

Given graded abelian groups B and C, we refer to a degree-preserving homomorphism $f : \Sigma^k B \to C$ as a *map of degree k from B to C*, and we write $|f| = k$. Thus in degree n, f takes B^{n-k} to C^n, therefore "increasing degree by k."

In this paper, suspensions are frequently omitted: the notation $B \to C$ does *not* necessarily mean a map of degree 0.

3. The spectral sequence

Let (C, d) be a chain complex. Assume that there is a short exact sequence of chain complexes in one of these forms (neglecting suspensions):

$$0 \to C \to C \to B \to 0,$$
$$0 \to C \to B \to C \to 0,$$
$$0 \to B \to C \to C \to 0.$$

Our goal is to set up a spectral sequence which "starts with the homology of B and converges to the homology of C."

Lemma 3.1. *Given a short exact sequence of any of the forms above, by replacing B and C with homotopy equivalent chain complexes, one can get short exact sequences of either of the following two forms:*

$$0 \to C \xrightarrow{\theta} C \to B \to 0, \tag{3.2}$$

$$0 \to B \to C \xrightarrow{\theta} C \to 0. \tag{3.3}$$

Proof. This is standard. □

Thus we may assume that there is a chain map $\theta : C \to C$ which is either injective or surjective, whichever happens to be convenient. The map θ need not have degree 0; we will allow its degree $|\theta|$ to be arbitrary. To keep the notation sparse, we will omit suspensions for now, writing $\theta : C \to C$ rather than $\theta : \Sigma^{|\theta|} C \to C$. We will provide full details on the grading when describing the E_1-term of the spectral sequence.

We write $H\theta : H^*C \to H^*C$ for the map induced by θ on homology.

To help in setting up the spectral sequence, and also with convergence, we have the following.

Criteria 3.4. We make one of the following two assumptions.

(a) For every $\alpha \in HC$, there is an n so that α is not in the image of $H\theta^n$, or
(b) for every $\alpha \in HC$, there is an n so that α is in the kernel of $H\theta^n$.

Criterion 3.4(a) says that nothing in HC is infinitely $H\theta$-divisible, while Criterion 3.4(b) says that everything in HC is $H\theta$-torsion. In the framework of the traditional Bockstein spectral sequence, these each avoid the presence of groups like $\mathbf{Q}$, and they also prevent the presence of groups like either $\mathbf{Q}/\mathbf{Z}_{(p)}$ (in case (a)) or $\mathbf{Z}_{(p)}$ (in case (b)). Either criterion also avoids the presence of summands like $\mathbf{Z}/q$ where q is prime to p. It is possible to consider the Bockstein spectral sequence when neither of these criteria holds, although convergence may be an issue then. Frequently, one or the other does hold, though, so with apologies, we omit the most general case. (This is meant to be a user's guide, not an exhaustive reference.)

3.1. **When Criterion 3.4(a) holds.** Assume that Criterion 3.4(a) is satisfied; then we use Lemma 3.1 to convert our short exact sequence into one of the form (3.2). We filter the complex C by setting $F^sC = \mathrm{im}(\theta^s)$, yielding

$$\cdots \subseteq F^2C \subseteq F^1C \subseteq F^0C = C. \tag{3.5}$$

Because of Criterion 3.4(a), the induced filtration on homology is Hausdorff. Because θ is injective, $F^sC \approx \Sigma^{s|\theta|}C$ for all s, and also $F^sC/F^{s+1}C \approx \Sigma^{s|\theta|}B$. Now take homology of this filtered chain complex; the Bockstein spectral sequence is the result. It has

$$\begin{aligned} E_1^{s,t} &= \begin{cases} H^{s+t}(F^sC/F^{s+1}C) \approx H^{s+t-s|\theta|}B & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases} \\ |d_r| &= (r, -r + |d|). \end{aligned} \tag{3.6}$$

The abutment is $H^{s+t}C$.

Notice that the E_1-term is made up of infinitely many copies of H^*B, one in each non-negative filtration, and this is not at all artificial: it follows naturally from the above filtration of C.

Notation 3.7. It will be useful to introduce notation to distinguish among the copies of H^*B in the spectral sequence. For $x \in H^nB$, we write $\overline{\theta}^s x$ for the corresponding class in $E_1^{s,n-s+s|\theta|} \approx H^nB$. More generally, because of the structure of differentials as explained below, the isomorphism $E_1^{0,n} \approx E_1^{s,n-s+s|\theta|}$ induces a surjection $E_r^{0,n} \twoheadrightarrow E_r^{s,n-s+s|\theta|}$ for each s, and given $x \in E_r^{0,n}$, we write $\overline{\theta}^s x$ for its image under this surjection.

We also mention that with respect to the isomorphism between $E_1^{*,*}$ and H^*B, the differential d_r sends elements from a subquotient of H^nB to a subquotient of $H^{n+|d|-r|\theta|}B$.

One nice feature of the Bockstein spectral sequence is that the differentials are periodic, and are determined by their effect on one edge of the spectral sequence.

In this situation, they are determined by their effect on elements in filtration 0. That is, for any $x \in E_r^{0,t}$, there is a nontrivial differential

$$d_r : x \longmapsto \overline{\theta}^r y$$

if and only if for all $i \geq 0$, the element $\overline{\theta}^i x$ is nonzero in E_r and there is a differential

$$d_r : \overline{\theta}^i x \longmapsto \overline{\theta}^{i+r} y.$$

This follows from the construction of the spectral sequence, and is easy to see if one unfolds the exact couple. This is also related to how differentials are computed, which should be familiar: to compute the rth differential on an element $x \in E_r^{0,t}$, lift x to a class in B^t, then to a class in C^t, and take its boundary. Since x has survived to the E_r-term, this boundary is in the image of θ^r, so apply $(\theta^r)^{-1}$. Project the result back to an element in H^*B; the result will represent a well-defined class in $E_r^{r,t-r+|d|}$.

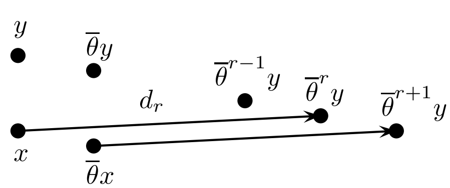
Furthermore, as the $\overline{\theta}$ notation suggests, the differentials determine the extensions in the abutment, reflecting the action of $H\theta$ on the homology of C. That is, if $x \in E_1^{0,n} \approx H^n B$ is an infinite cycle and not a boundary in the spectral sequence, then it survives to an "$H\theta$-periodic" element x in $H^n C$:

- The elements $\overline{\theta}^i x$ survive for all $i \geq 0$.
- The element $x \in H^n B$ lifts to an element $\overline{x} \in H^n C$.
- For each $i \geq 0$, the element $\overline{\theta}^i x$ lifts to the element $(H\theta)^i \overline{x} \in H^{n+i|\theta|} C$.

The presence of a differential $d_r : x \mapsto y$ from a class $x \in E_r^{0,n}$ to $\overline{\theta}^r y \in E_r^{r,n+|d|-r}$ means that r "copies" of y survive to E_∞: the elements $\overline{\theta}^i y$ survive for $0 \leq i \leq r-1$. In this case, $y \in H^{n+|d|-r|\theta|} B$ lifts to an element $\overline{y} \in H^{n+|d|-r|\theta|} C$, and the elements $\overline{\theta}^i y$ correspond in the abutment to the finite $H\theta$-family

$$\{\overline{y}, (H\theta)\overline{y}, \ldots, (H\theta)^{r-1}\overline{y}\} \subseteq H^* C.$$

Here is a picture:



In the language of Boardman [Boa99], this is a half-plane spectral sequence with entering differentials, so by [Boa99, Theorem 7.1], this spectral sequence converges conditionally. It converges strongly with some additional hypotheses, the most general of which is that RE_∞^s, defined in [Boa99, 5.1], is zero. Some hypotheses which imply this, and which are easier to verify, are listed in the theorem below, which summarizes the properties of the spectral sequence.

Theorem 3.8. *Assume that we have a short exact sequence of chain complexes*

$$0 \to \Sigma^{|\theta|} C \xrightarrow{\theta} C \to B \to 0,$$

and assume that Criterion 3.4(a) is satisfied. Then there is a spectral sequence, the Bockstein spectral sequence*, which has the following properties.*

(a) *(E_1-term and grading) It has*

$$E_1^{s,t} \approx \begin{cases} H^{s+t-s|\theta|}B & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases}$$
$$|d_r| = (r, -r + |d|).$$

(b) *(Convergence) It converges conditionally to $H^{s+t}C$. If $|\theta| > 0$ and H^*B is bounded below, then the spectral sequence converges strongly. If for each s and t, there is an r so that $E_r^{s,t}$ is finite, then the spectral sequence converges strongly.*

(c) *(Periodicity of the differentials) With notation as in Notation 3.7, for any $x \in E_r^{0,t}$, there is a differential*

$$d_r : x \longmapsto \overline{\theta}^r y$$

if and only if for any $i \geq 0$, there is a differential

$$d_r : \overline{\theta}^i x \longmapsto \overline{\theta}^{i-r} y.$$

(d) *(Extensions and interpretation of the differentials) The differentials determine the extensions, reflecting the action of $H\theta$ on the homology of C:*

(i) *If $x \in E_1^{0,n} \approx H^nB$ is an infinite cycle and not a boundary in the spectral sequence, then it survives to an $H\theta$-periodic family $\{(H\theta)^i\overline{x} : i \geq 0\}$ in H^*C, where $\overline{x} \in H^nC$ is a lift of $x \in H^nB$.*

(ii) *The presence of a differential d_r from a class $x \in E_r^{0,n}$ to $\overline{\theta}^r y \in E_r^{r,n+|d|-r}$ means that in the abutment, there is a finite $H\theta$-family*

$$\{\overline{y}, (H\theta)\overline{y}, \ldots, (H\theta)^{r-1}\overline{y}\} \subseteq H^*C,$$

where $\overline{y} \in H^{n+|d|-r|\theta|}C$ is a lift of $y \in H^{n+|d|-r|\theta|}B$.

3.2. **When Criterion 3.4(b) holds.** Assume that Criterion 3.4(b) is satisfied. Then we use Lemma 3.1 to convert our short exact sequence into one of the form (3.3). We filter the complex C by setting $F_sC = \ker(\theta^{s+1})$, yielding

$$0 = F_{-1}C \subseteq F_0C \subseteq F_1C \subseteq \cdots \subseteq C.$$

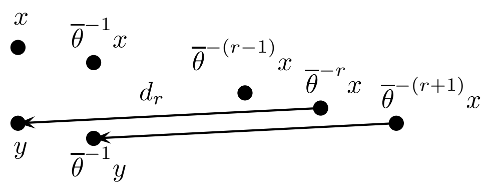
Criterion 3.4(b) says that the induced filtration exhausts H^*C. For each s, $F_sC/F_{s-1}C \approx \Sigma^{s|\theta|}B$. Now take homology of this filtered chain complex; the Bockstein spectral sequence is the result. It has

$$E_1^{s,t} = \begin{cases} H^{s+t}(F_sC/F_{s-1}C) \approx H^{s+t-s|\theta|}B & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases} \tag{3.9}$$
$$|d_r| = (-r, r + |d|).$$

The abutment is $H^{s+t}C$. As in the previous case, the E_1-term is made up of infinitely many copies of H^*B. Now, though, for $x \in H^sB$, we write $\overline{\theta}^{-s}x$ for the corresponding class in $E_1^{s,t-s+s|\theta|} \approx H^sB$: in this situation, the sth filtration corresponds to division by θ^s. Because of this, some details are different in this version of the spectral sequence, but the main ideas are the same.

One difference here is that this is a half-plane spectral sequence with exiting differentials, and so [Boa99, Theorem 6.1] says that the spectral sequence converges strongly.

The differentials are periodic in this case as well, but in a slightly different way, and the effect of the differentials on the abutment is also slightly different. Here is a picture of the differentials in this spectral sequence:



We summarize the spectral sequence's properties.

Theorem 3.10. *Assume that we have a short exact sequence of chain complexes*

$$0 \to B \to \Sigma^{|\theta|}C \xrightarrow{\theta} C \to 0,$$

and assume that Criterion 3.4(b) is satisfied. Then there is a spectral sequence, the Bockstein spectral sequence*, which has the following properties.*

(a) *(E_1-term and grading) It has*

$$E_1^{s,t} \approx \begin{cases} H^{s+t-s|\theta|}B & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases}$$
$$|d_r| = (-r, r+|d|).$$

(b) *(Convergence) It converges strongly to $H^{s+t}C$.*

(c) *(Periodicity of the differentials) With notation as above, for any $\overline{\theta}^{-r}x \in E_r^{r,t-r+|d|}$, there is a differential*

$$d_r : \overline{\theta}^{-r}x \longmapsto y \in E_r^{0,t}$$

if and only if for any $i \geq 0$, there is a differential

$$d_r : \overline{\theta}^{-r-i}x \longmapsto \overline{\theta}^{-i}y.$$

(d) *(Extensions and interpretation of the differentials) The differentials determine the extensions, reflecting the action of $H\theta$ on the homology of C:*

   (i) *If $x \in E_1^{0,n} \approx H^nB$ is an infinite cycle and not a boundary in the spectral sequence, then it survives to an infinitely $H\theta$-divisible family: a family*

   $$\{(H\theta)^{-i}\overline{x} \in H^{n-i|\theta|}C \,:\, i \geq 0\},$$

   where $\overline{x} \in H^nC$ is a lift of $x \in H^nB$.

   (ii) *The presence of a differential d_r from a class $\overline{\theta}^{-r}x \in E_r^{r,n}$ to $y \in E_r^{0,n+|d|}$ means that in the abutment, there is a finite $H\theta$-family*

   $$\{\overline{x}, (H\theta)^{-1}\overline{x}, \ldots, (H\theta)^{-(r-1)}\overline{x}\} \subseteq H^*C,$$

   where $\overline{x} \in H^nC$ is a lift of $x \in H^nB$.

4. Examples

We briefly discuss some examples.

First, we consider the classical case in cohomology (see [McC01, Section 10.1] or [Boa99, Section 14], for example). Let (C, d) be a chain complex of free abelian groups with $|d| = 1$, and suppose we want to "compute H^*C from $H^*(C \otimes \mathbf{Z}/p)$." Suppose that the cohomology groups of C are all finitely generated. We need to eliminate any summands in H^*C of the form $\mathbf{Z}/q$ where q is prime to p, because they are invisible in $H^*(C \otimes \mathbf{Z}/p)$, so we replace C by its p-localization $C_{(p)} = C \otimes \mathbf{Z}_{(p)}$. Now consider the short exact sequence

$$0 \to C_{(p)} \xrightarrow{p} C_{(p)} \to C \otimes \mathbf{Z}/p \to 0.$$

We are exactly in the situation of Theorem 3.8. We get a spectral sequence with

$$\begin{aligned} E_1^{s,t} &\approx \begin{cases} H^{s+t}(C \otimes \mathbf{Z}/p) & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases} \\ |d_r| &= (r, -r+1). \end{aligned} \tag{4.1}$$

It converges strongly to $H^{s+t}C_{(p)}$, because $E_1^{s,t}$ is finite for each s and t. An infinite cycle in $E_1^{0,t} \approx H^tB$ corresponds to a copy of $\mathbf{Z}_{(p)}$ in $H^tC_{(p)}$, and a nonzero differential $d_r : E_r^{0,t} \to E_r^{r,t-r+1}$ corresponds to a copy of $\mathbf{Z}/(p^r)$ in $H^{t+1}C_{(p)}$. Since one can recover the p-part of H^*C from $H^*C_{(p)}$, this justifies the statement in the introduction that one can set up the Bockstein spectral sequence so that "the abutment is essentially the homology of the original chain complex."

For the classical case in homology, the same approach can be used, but then one gets differentials with a somewhat unusual grading: d_r has bidegree $(r, -r-1)$. To repair this, one can regrade the filtration (3.5) defining the spectral sequence by setting $F'_s = F^{-s}$, which results in a spectral sequence with

$$\begin{aligned} E^1_{s,t} &= \begin{cases} H_{s+t}(F'_sC/F'_{s-1}C) \approx H_{s+t}(C \otimes \mathbf{Z}/p) & \text{if } s \leq 0, \\ 0 & \text{if } s > 0, \end{cases} \\ |d^r| &= (-r, r-1). \end{aligned} \tag{4.2}$$

Viewed this way, this is a left half-plane spectral sequence. The differentials have an interpretation similar to the cohomological case. If the homology groups of C are finitely generated, then the spectral sequence converges strongly to $H_*C_{(p)}$.

A non-classical Bockstein spectral sequence arises on [MRW77, p. 310], and they allude to convergence in their Remark 3.11.

This author used a non-classical Bockstein spectral sequence in [Pal, Section 3]. It satisfies Criterion 3.4(a); it is trigraded and each trigraded piece is finite, so it converges strongly by Theorem 3.8(b).

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