Algebraic topology in Sage

John H. Palmieri

Department of Mathematics
University of Washington

Cascade Topology Seminar
December 1, 2012
Eugene

Sage

Sage's mission: Creating a viable free open source alternative to Magma™, Maple™, Mathematica™, and Matlab™. In detail:

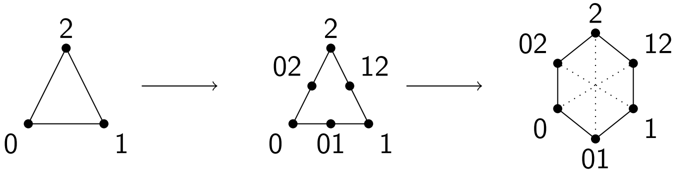
- free
- free
- open-source
- open-source

Sage project started by William Stein in 2005.

Sage in action...

Triangulating $\mathbf{R}P^n$

- Start with $\partial\Delta^{n+1}$ (the n-sphere).
- Barycentrically subdivide once to get a complex K
- Each vertex of K corresponds to a nonempty proper subset of $\{0, 1, 2, \ldots, n+1\}$.
- Taking complements gives the simplicial antipodal map on K; the quotient space by the resulting $\mathbf{Z}/2$-action is a triangulation of $\mathbf{R}P^n$.



Triangulating $\mathbf{R}P^n$

This is usually not a *minimal* triangulation of $\mathbf{R}P^n$. Minimal number of vertices required to triangulate $\mathbf{R}P^n$:

- $n = 2$: 6 vertices
- $n = 3$: 11 vertices
- $n = 4$: 16 vertices
- $n = 5$: 22 vertices? (unpublished example with 24 vertices)
- $n \geq 3$: need at least $1 + (n+1)(n+2)/2$ vertices. Bound not known to be optimal for $n \geq 5$.
- The above construction uses $2^{n+1} - 1$ vertices.

Triangulating $\mathbf{C}P^n$

$\mathbf{C}P^n$:

- $\mathbf{C}P^2$ was first triangulated (minimally – 9 vertices) by Kühnel and Banchoff in 1983.
- $\mathbf{C}P^3$ was first triangulated in 2010 (!). Triangulation uses 18 vertices.
- Need at least $1 + (n+1)^2/2$ vertices for $\mathbf{C}P^n$. Bound not known to be optimal for $n \geq 3$.

Problem

No known triangulation of $\mathbf{C}P^n$ for $n \geq 4$.

Torsion in homology of simplicial complexes

Problem

Given an integer $n > 0$, what kinds of torsion can arise in the homology of a simplicial complex on n vertices?

Start by looking at *random simplicial complexes* (Meshulam and Wallach):

- Fix $n =$ number of vertices, $d =$ dimension, and $p =$ probability.
- Include all simplices of dimension $< d$.
- Include each d-simplex independently with probability p.

Torsion in random simplicial complexes

# vert.	dim.	max torsion (for best probability p)
8	2	10000 trials: 3 complexes with 2-torsion
10	2	10000 trials: 15 complexes with 2-torsion
10	3	10000 trials: 41 with 2-torsion, 2 with 3-torsion, 1 with 4-torsion
10	4	10000 trials: 64 with 2-torsion, 2 with 3-torsion
13	2	10000 trials: 27 with 2-torsion, 1 with 3-torsion
13	4	2000 trials: 62 with 2-torsion, 8 with 3-torsion, 6 with 4-torsion, 2 each with 10-, 14-torsion, 1 each with 8-, 11-, 12-, 18-, 22-, 30-torsion
20	2	10000 trials: 93 with 2-torsion, 5 with 3-torsion 3 with 4-torsion, 1 with 5-torsion, 1 with 7-torsion
30	2	10000 trials: 113 with 2-torsion, 17 with 3-torsion, 8 with 4-torsion, 2 with 5-torsion, 4 with 6-torsion, 1 each with 7-, 9-, 36-torsion

Sum complexes – Linial, Meshulam, Rosenthal

- Fix $n > 0$. Vertex set $= \mathbf{Z}/n$.
- Fix dimension $d > 0$ and subset A of $\mathbf{Z}/n$ with cardinality $d + 1$.
- The facets are

$$\{(x_0, x_1, \ldots, x_d) : \sum x_i \in A\}.$$

- Let's compute some homology groups. . .

Torsion: problems

For any finitely generated abelian group G, let max-cyclic(G) be the order of the largest cyclic summand of G. For any integer n, define $T(n)$ by

$$T(n) = \max\{\text{max-cyclic}(H_* X) : X \text{ has } n \text{ vertices}\}.$$

It seems that $T(n)$ grows more quickly than exponential in n.

Problem

- *Understand the asymptotic behavior of $T(n)$.*
- *Compute $T(n)$ for each n.*
- *The same but fixing a dimension d as well as n:*

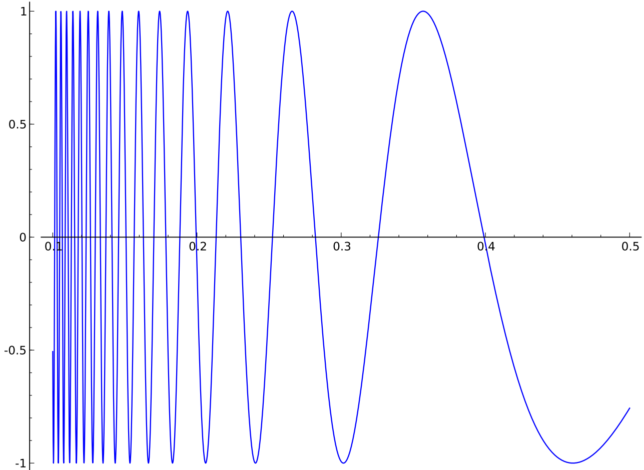
$$T(n,d) = \max\{\textit{max-cyclic}(H_{d-1} X) : X \textit{ has } n \textit{ vertices}\}.$$

SageTeX

Advertisement: SageTeX. In a LaTeX file (like this one),

```
\sageplot{plot(sin(1/x^2), (x, 0.1, 0.5))}
```

yields



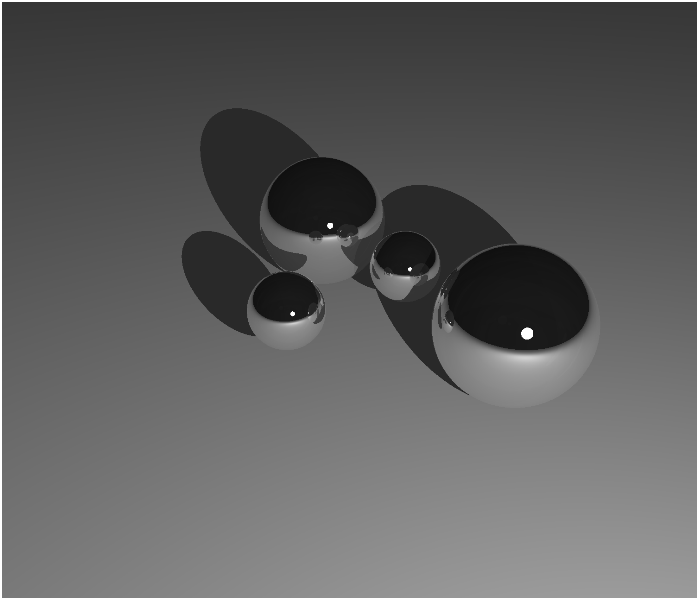
SageTEX

```
\begin{sagesilent}
t = Tachyon(camera_center=(8.5,5,5.5), look_at=(2,0,0), raydepth=6,
             xres=1500, yres=1500)
t.light((10,3,4), 1, (1,1,1))
t.texture('mirror', ambient=0.05, diffuse=0.05, specular=.9,
             opacity=0.9, color=(.8,.8,.8))
t.texture('grey', color=(.8,.8,.8), texfunc=0) ## try other values of texfunc!
t.plane((0,0,0),(0,0,1),'grey')
t.sphere((4,-1,1), 1, 'mirror')
t.sphere((0,-1,1), 1, 'mirror')
t.sphere((2,-1,1), 0.5, 'mirror')
t.sphere((2,1,1), 0.5, 'mirror')
\end{sagesilent}

\sageplot{t}
```

yields

SageTEX



Conclusion

Main Sage web site: http://www.sagemath.org

Also look into http://aleph.sagemath.org and the Sage app for smartphones.

I will end with http://www.nilesjohnson.net/hopf.html. Every frame of this movie was made using Sage, and then the frames were animated with FFmpeg.