Cohomology operations and the Steenrod algebra

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WCATSS, 27 August 2011
Cohomology operations

cohomology operations = NatTransf($H^n(\_, G), H^m(\_, G')$).
If $X$ is a CW complex, then

$$H^n(X; G) \cong [X, K(G, n)].$$

So by Yoneda’s lemma, there is a bijection

$$\text{NatTransf}(H^n(\_, G), H^m(\_, G')) 
\leftrightarrow [K(G, n), K(G', m)] 
\cong H^m(K(G, n); G').$$

Thus elements of $H^m(K(G, n); G')$ give cohomology operations.
Serre, Borel, Cartan, et al. computed the groups $\tilde{H}^m(K(G, n); G')$ for $G, G'$ finite abelian.

First, note that they’re zero when $m < n$ (by the Hurewicz theorem).

Now focus on case $G = \mathbb{Z}/p\mathbb{Z} = G'$, with $p$ a prime.

The groups stabilize: for all $q, n$, there is a map

$$H^{q+n}(K(\mathbb{Z}/p\mathbb{Z}, n); \mathbb{Z}/p\mathbb{Z}) \rightarrow H^{q+n-1}(K(\mathbb{Z}/p\mathbb{Z}, n-1); \mathbb{Z}/p\mathbb{Z}),$$

It’s an isomorphism when $q < n - 1$.

Iterate this. The inverse limit is the collection of mod $p$ stable cohomology operations of degree $q$. Assemble together for all $q$: you get the mod $p$ Steenrod algebra, which is an $\mathbb{F}_p$-algebra under composition.
The mod 2 Steenrod algebra $A$

- For any space (or spectrum) $X$, $H^*(X; F_2)$ is a module over $A$.
- $A$ is generated as an algebra by elements $Sq^q$ (pronounced “square $q$”), with $Sq^q : H^n(\_\_) \to H^{n+q}(\_\_)$.
- If $X$ is a space:
  - $Sq^q : H^q X \to H^{2q} X$ is the cup-squaring map.
  - $Sq^q : H^i X \to H^{i+q} X$ is zero if $i < q$.
- $A$ is associative, non-commutative. (Example: $Sq^1 Sq^2 \neq Sq^2 Sq^1$. On the polynomial generator $x$ of $H^*(\mathbb{R}P^\infty)$, $Sq^2 Sq^1(x) = x^4$ while $Sq^1 Sq^2(x) = 0$.)
Applications

- Two spaces can have the same cohomology rings but different module structures over the Steenrod algebra, in which case they can’t be homotopy equivalent. (Example: \( \Sigma \mathbb{C}P^2 \) and \( S^3 \vee S^5 \).)

- The Hopf invariant one problem: a nice multiplication on \( \mathbb{R}^n \) \( \rightsquigarrow \) a CW complex with mod 2 cohomology

\[
\begin{array}{ccc}
\bullet & x & x^2 \\
1 & \text{dim } n
\end{array}
\]

Hence \( \text{Sq}^n(x) \neq 0 \) while \( \text{Sq}^i(x) = 0 \) for \( 0 < i < n \). Thus \( \text{Sq}^n \) must be indecomposable in the mod 2 Steenrod algebra. This implies that \( n \) is a power of 2. (Adams refined this approach to solve the problem completely: \( n = 1, 2, 4, 8 \).)

- See Mosher-Tangora for more details and examples.
The Adams spectral sequence

Fix a prime $p$ and let $A$ be the mod $p$ Steenrod algebra. For spaces or spectra $X$ and $Y$, there is a spectral sequence, the Adams spectral sequence, with

$$E_2 \cong \text{Ext}_A^*(H^*Y, H^*X) \Rightarrow [X, Y].$$

It “converges” if $X$ and $Y$ are nice enough.
Other topics:

- $A$ is a graded Hopf algebra.
- Milnor’s theorem: the graded vector space dual $A_\ast$ of $A$ has a very nice structure. At the prime 2: as algebras, $A_\ast \cong F_2[\xi_1, \xi_2, \xi_3, \ldots]$, and there is a simple formula for the comultiplication on each $\xi_n$.
- You can do computations in $A$ using Sage.
- For generalized homology theories, it is often better to work with homology rather than cohomology: if $E$ is a spectrum representing a homology theory, then $E_* E$ is often better behaved than $E^* E$.
- For spectra $X$ and $Y$, the Adams spectral sequence looks like

$$E_2 \cong \text{Ext}^*_E(E_* X, E_* Y),$$

abutting to $[X, Y]$. 
Fix a prime $p$ and let $A$ be the mod $p$ Steenrod algebra. Mod $p$ cohomology defines a functor

$$\text{Spectra}^{\text{op}} \to A\text{-Mod}.$$ 

Make a new category, $A\tilde{\text{Mod}}$: same objects as $A\text{-Mod}$, but the morphisms from $M$ to $N$ are $\text{Ext}_A^*(M, N)$. Then we have functors

$$\text{Spectra}^{\text{op}} \to A\text{-Mod} \to A\tilde{\text{Mod}}$$

as well as a connection, via the Adams SS,

$$A\tilde{\text{Mod}} \leadsto \text{Spectra}^{\text{op}}.$$ 

So via cohomology and the Adams SS, the category $A\tilde{\text{Mod}}$ is an approximation to the category of spectra.
Furthermore, $\widetilde{A-\text{Mod}}$ (actually a “fattened up” version of this category) has many formal similarities to Spectra: it satisfies the axioms for a stable homotopy category.

Some details:

- $A_* = \text{graded dual of the mod } p \text{ Steenrod algebra}$.
- $\text{Ch}(A_*) = \text{category with objects cochain complexes of } A_*\text{-comodules, morphisms cochain maps}$. Then $\text{Ch}(A_*)$ has a (cofibrantly generated) model category structure.
- Cofibrations: degree-wise monomorphisms. Fibrations: degree-wise epimorphisms with degree-wise injective kernel. Weak equivalences: maps $f : X \to Y$ which induce an isomorphism

$$[\Sigma^i F_p, J \otimes X] \to [\Sigma^i F_p, J \otimes Y],$$

where $J$ is an injective resolution of the trivial module $F_p$. 
The associated homotopy category is a stable homotopy category. Call it Stable($A_\ast$).

**Alternative construction**

Stable($A_\ast$) is the category with objects cochain complexes of injective $A_\ast$-comodules, morphisms cochain homotopy classes of maps.

- Smash product: $- \otimes_{F_p} -$  
- Sphere object: injective resolution of $F_p$
There is a functor, in fact the inclusion of a full subcategory,

\[ A_\ast\text{-Comod} \to A\text{-Mod}. \]

If \( M \) and \( N \) are \( A_\ast \)-comodules, then

\[ \text{Hom}_{\text{Stable}(A_\ast)}(\Sigma^i M, N) \cong \text{Ext}_{A}^{i}(M, N). \]

So cohomology gives us a functor

\[ \text{Spectra} \to \text{Stable}(A_\ast) \]

and the Adams spectral gives a loose connection

\[ \text{Stable}(A_\ast) \rightsquigarrow \text{Spectra}. \]

Furthermore, \( \text{Stable}(A_\ast) \) is a stable homotopy category.
Theorem (Nishida’s theorem)

If \( n > 0 \), then every \( \alpha \in \pi_n(S^0) \) is nilpotent.

Analogue for the Steenrod algebra – things are more complicated:

Theorem

Let \( p = 2 \). There is a ring \( R \) and a ring map \( \Ext^*_A(F_2, F_2) \to R \) which is an isomorphism mod nilpotence.

The ring \( R \) can be described explicitly.

(This is also analogous to the Quillen stratification theorem for group cohomology.)
Idea of proof.

- Stable($A_*$) is a stable homotopy category and $F_2$ is the sphere object. So $\text{Ext}^*_A(F_2, F_2)$ is the “homotopy groups of spheres”.
- So there are Adams SS converging to $\text{Ext}^*_A(F_2, F_2)$.
- That is:
  \[ ? \implies \text{Ext}^*_A(F_2, F_2) \implies \pi_*(S^0) \]
- In particular: any Lyndon-Hochschild-Serre spectral sequence can be viewed as an Adams spectral sequence.
- For a certain normal sub-Hopf algebra $D \leq A$:
  \[ \text{Ext}^*_A(F_2, F_2) \to R \text{ is (essentially) an edge homomorphism.} \]
- Use properties of Adams spectral sequences, plus some computations, to show that, mod nilpotence, $R$ detects everything in $\text{Ext}^*_A(F_2, F_2)$. 

Lemma

Vanishing lines in Adams spectral sequences are generic.

That is: For fixed ring spectrum $E$ and fixed slope $m$, the collection of all objects $X$ for which, at some term of the Adams spectral sequence

$$\text{Ext}_{E^*E}(E^*, E_*X) \Rightarrow \pi_*X,$$

there is a vanishing line of slope $m$, is a thick subcategory.

To prove the theorem, show that for each $m > 0$, there is a vanishing line of slope $m$ at some term of the spectral sequence

$$\text{Ext}_{A//D}^*(F_2, \text{Ext}_D^*(F_2, F_2)) \Rightarrow \text{Ext}_A^*(F_2, F_2).$$

This implies that everything not on the bottom edge is nilpotent...
This Steenrod analogue of Nishida’s theorem can be modified to give a nilpotence theorem (à la DHS).

**Question**

What about a thick subcategory theorem?

**Question**

What about the Bousfield lattice?

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What about a periodicity theorem and other chromatic structure?
Other ideas:

- Study Bousfield localization in $\text{Stable}(A_*)$.
- Investigate the telescope conjecture.
- Investigate the odd primary case.
- Investigate analogous categories coming from $BP_*BP$ or other Hopf algebroids.
References


