## 1 $\Delta$-complexes

**Definition 1.1.** Let $\Delta^n$ be the *standard $n$-simplex*, consisting of the points $(x_0, x_1, \ldots, x_n) \in \mathbf{R}^{n+1}$ satisfying $\sum x_i = 1$, $0 \leq x_i \leq 1$ for all $i$. There are *coface maps* $d^i : \Delta^{n-1} \to \Delta^n$, $0 \leq i \leq n$, defined by

$$d^i(x_0, x_1, \ldots, x_{n-1}) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}).$$

Note that the coface maps satisfy identities:

$$d^j d^i = d^i d^{j-1} \text{ for all } i < j.$$

**Definition 1.2.** An *abstract $\Delta$-complex* $X$ consists of a collection of sets $X_n$, $n \geq 0$, together with functions $d_i : X_n \to X_{n-1}$, $0 \leq i \leq n$, satisfying the *simplicial identity*

$$d_i d_j = d_{j-1} d_i \text{ for all } i < j.$$

The elements of $X_n$ are called *$n$-simplices.* The maps $d_i$ are called *face maps.*

Given an abstract $\Delta$-complex $X$, its *geometric realization* is the following topological space $|X|$:

$$|X| = \coprod_n X_n \times |\Delta^n| / \sim = \coprod_{n, x \in X_n} (\Delta^n, x) / \sim,$$

where $(z, d_i x) \sim (d^i z, x)$ for $x \in X_n$, $z \in \Delta^{n-1}$. In the right-hand disjoint union, you should interpret "$(\Delta^n, x)$" as being the copy of $\Delta^n$ labeled by $x \in X_n$.