Math 136: Homework 6

Due Thursday, May 17

1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and let I be an open interval of the form $I = (a, b)$ where $a < b$. Using the definition of continuity, show that the *preimage* of I, denoted $f^{-1}(I)$ and defined by
$$f^{-1}(I) = \{\mathbf{x} \in \mathbb{R}^n \,:\, f(\mathbf{x}) \in I\},$$
is an open set. (The function f is not assumed to be invertible: $f^{-1}(I)$ is notation for the preimage of the set I.)

2. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by
$$g(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0), \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$
Show that
$$g_y(x, 0) = \lim_{h \to 0} \frac{g(x, h) - g(x, 0)}{h} = x$$
and similarly that $g_x(0, y) = -y$. Hence, show that $g_{yx}(0, 0) = 1$ and $g_{xy}(0, 0) = -1$.

3. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2-function (that is, the second partial derivatives of f are all continuous). Define $g : \mathbb{R} \to \mathbb{R}$ by the formula
$$g(t) = f(\mathbf{x} + t\mathbf{h})$$
where $\mathbf{x}$ and $\mathbf{h}$ are vectors in $\mathbb{R}^2$. Use the chain rule to find formulae for both $g'(t)$ and $g''(t)$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Suppose that $M > 0$ is a real number such that $|f_{xx}| \leq M$, $|f_{xy}| \leq M$, and $|f_{yy}| \leq M$. Use problem (3) above and formula (12.6.3) of Salas-Hille-Etgen (page 605) to show that
$$|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - \nabla f(\mathbf{x}) \cdot \mathbf{h}| \leq M \|\mathbf{h}\|^2 .$$