Math 136: Homework 5

Due Thursday, May 3

1. Consider the linear map $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathbf{v} \mapsto A\mathbf{v}$, where $A = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 4 & 0 \\ -2 & -2 & 1 \end{pmatrix}$. Find the matrix associated to T_A with respect to the basis

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

2. Consider the function

$$f(x, y) = 3x^2 + 2xy + 3y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Consider the new variables (X, Y) where

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

For an appropriate choice of θ, the function f will assume the form $f(x, y) = aX^2 + bY^2$. Find a, b, and θ by diagonalizing the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ and show that $e^A = \begin{pmatrix} e & e^2 - e & e^3 - e^2 \\ 0 & e^2 & e^3 - e^2 \\ 0 & 0 & e^3 \end{pmatrix}$.

4. Let I denote the $n \times n$ identity matrix, and suppose that B is an $n \times n$ matrix satisfying $B^2 = 0$ (for example, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$). Let c be a scalar and show that $e^{cI+B} = e^c I + e^c B$. (Recall from class that if X and Y are matrices which commute, then $e^{X+Y} = e^X e^Y$.)