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Suppose that $f(x)$ has $n+1$ continuous derivatives, and let $P_n(x)$ be the nth Taylor polynomial of x about $a = 0$. The estimate for the remainder $R_{n+1}(x) = f(x) - P_n(x)$ on p. 605 of Salas-Hille-Etgen (12.6.3) can be restated as follows:

> If $\left|f^{(n+1)}(x)\right| \le C$ for all x in some interval J containing 0, then $|R_n(x)| \le \dfrac{C|x|^{n+1}}{(n+1)!}$ for all $x \in J$.

Definition 1 ("Big O" notation)**.** If $g(x)$ is a function defined near $x = 0$, and if there is a constant C such that $|g(x)| \le C|x|^k$ for x near 0, then we say that $g(x)$ is $O(x^k)$ (as $x \to 0$).

With this notation, we have $R_n(x) = O(x^{n+1})$, or

$$f(x) = P_n(x) + O(x^{n+1}) \quad \text{as } x \to 0. \tag{2}$$

Moreover, $P_n(x)$ is the only polynomial of degree at most n with this property.

Proposition 3. *Suppose that $f(x)$ has $n+1$ continuous derivatives, and suppose that $Q_n(x)$ is a polynomial of degree at most n such that $f(x) = Q_n(x) + O(x^{n+1})$ as $x \to 0$. Then $Q_n(x) = P_n(x)$.*

Proof. Subtract the equation in the statement from $f(x) = P_n(x) + O(x^{n+1})$ to get $P_n(x) - Q_n(x) = O(x^{n+1})$. Let $P_n(x) = \sum_{k=0}^{n} a_n x^n$ and $Q_n(x) = \sum_{k=0}^{n} b_n x^n$; then we have

$$(a_0 - b_0) + (a_1 - b_1)x + \cdots + (a_n - b_n)x^n = O(x^{n+1}). \tag{4}$$

Plugging in $x = 0$ gives $a_0 - b_0 = 0$, so $a_0 = b_0$. So cancel those terms from (4) and divide by x:

$$(a_1 - b_1) + (a_2 - b_2)x + \cdots + (a_n - b_n)x^{n-1} = O(x^n).$$

Set $x = 0$ again to get $a_1 = b_1$. Continue inductively to find that $a_k = b_k$ for all k, which means that $P_n(x) = Q_n(x)$. □

Proposition 3 is useful for calculating Taylor polynomials: if we can use any method at all to find a polynomial $Q_n(x)$ of degree at most n so that $f(x) = Q_n(x) + O(x^{n+1})$, then $Q_n(x)$ must equal $P_n(x)$. Here are two applications.

Taylor polynomials and l'Hôpital's rule.

Suppose that f, g, and their first $k-1$ derivatives vanish at $x = 0$, but $g^{(k)}(0)$ does not vanish. The Taylor expansions of f and g then look like

$$f(x) = \frac{f^{(k)}(0)}{k!}x^k + O(x^{k+1}), \quad g(x) = \frac{g^{(k)}(0)}{k!}x^k + O(x^{k+1}).$$

Taking the quotient and canceling out $x^k/k!$ gives

$$\frac{f(x)}{g(x)} = \frac{f^{(k)}(0) + O(x)}{g^{(k)}(0) + O(x)} \to \frac{f^{(k)(0)}}{g^{(k)}(0)} \quad \text{as } x \to 0.$$

This is just what l'Hôpital's rule says, but we can sometimes use the earlier observation to compute the answer without computing all of the derivatives.

Example 5. What is

$$\lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}?$$

Since

$$\sin^2 x = \left(x - \frac{x^3}{6} + O(x^5)\right)^2 = x^2 - \frac{x^4}{3} + O(x^6),$$

we get $x^2 \sin^2 x = x^4 + O(x^6)$ and

$$\frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \frac{\frac{1}{3}x^4 + O(x^6)}{x^4 + O(x^6)} = \frac{\frac{1}{3} + O(x^2)}{1 + O(x^2)} \to \frac{1}{3}.$$

Example 6. What is

$$\lim_{x\to 1} \left(\frac{1}{\log x} + \frac{x}{x-1}\right)?$$

Since the limit is as $x \to 1$, we need to expand Taylor series about 1. First of all,

$$\frac{1}{\log x} + \frac{x}{x-1} = \frac{x - 1 - x\log x}{(x-1)\log x} = \frac{(x-1) - (x-1)\log x - \log x}{(x-1)\log x}.$$

Next, if we expand $\log x$ about $a = 1$, we get $\log x = (x-1) + \frac{1}{2}(x-1)^2 + O((x-1)^3)$, and plugging this in yields

$$\frac{(x-1) - (x-1)^2 - \left[(x-1) - \frac{1}{2}(x-1)^2\right] + O((x-1)^3)}{(x-1)^2 + O((x-1)^3)} = \frac{-\frac{1}{2} + O(x-1)}{1 + O(x-1)} \to -\frac{1}{2}.$$

Higher derivative tests for critical points.

Recall that if $f'(a) = 0$, then $f(x)$ has a local maximum at $x = a$ if $f''(a) < 0$, and similarly it has a local minimum if $f''(a) > 0$. What happens if $f''(a) = 0$? Then the behavior of f near a is controlled by the first nonvanishing derivative at a.

Proposition 7. *Suppose that $f(x)$ has k continuous derivatives near a, with $f'(a) = f''(a) = \cdots = f^{(k-1)}(a) = 0$, but $f^{(k)}(a) \neq 0$. If k is even, then f has a local max if $f^{(k)}(a) < 0$, while it has a local min if $f^{(k)}(a) > 0$. If k is odd, it has neither a local max nor a local min.*

Proof. The degree $k-1$ Taylor polynomial for $f(x)$ about $x=a$ is simply the constant $f(a)$ – all the other terms are zero. So Taylor's formula of order $k-1$ with remainder becomes

$$f(x) = f(a) + \frac{f^{(k)}(c)}{k!}(x-a)^k \quad \text{for some } c \text{ between } x \text{ and } a.$$

If x is close to a, then so is c, so $f^{(k)}(c)$ is close to $f^{(k)}(a)$, by continuity of f^k. In particular, it is nonzero, with the same sign as $f^{(k)}(a)$. Furthermore, $(x-a)^k$ is always non-negative if k is even, but it changes sign at $x=a$ if k is odd. Thus if k is even, $f(x)-f(a)$ always has the same sign – the sign of $f^{(k)}(a)$ – when x is near a. For example, if k is even and $f^{(k)}(a)$ is negative, then $f(x)-f(a)<0$ for all x near a: that is, $f(x)<f(a)$ for all x near a, which means that $f(a)$ is a local maximum. On the other hand, if k is odd, then $f(x)-f(a)$ changes sign at $x=a$, so $f(a)$ is neither a max nor a min. □