Four old Math 135 midterms

(A1) Evaluate $\lim_{x\to 0} \dfrac{x\sin(x^2)-\sin(x^3)}{\sin(x^7)}$ in any way you wish.

(A2) Evaluate the integral $\displaystyle\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx$ or explain why you can't.

(A3) Consider the power series $\displaystyle\sum_{k=1}^{\infty} \frac{2^k \ln(k+1)}{k} x^k$.

(a) Find its radius of convergence.

(b) Find its interval of convergence.

(c) For what values of x the series absolutely convergent? For what values of x is the series conditionally convergent?

(A4) Consider the sequence $\{a_k\}$ defined by $a_0 = 0$, $a_{n+1} = 1 + m\,a_n$, where m is a real number with $|m| < 1$. Does the sequence converge? Explain your answer. If the sequence does converge, what is its limit?

(A5) Find the first three non-zero terms in the series expansion of $\arcsin(x)$ about $x = 0$.

Hint: *Recall that* $\arcsin(x) = \int_0^x (1-t^2)^{-1/2} dt$. *If you wish, you may use the binomial expansion of* $(1+x)^{-1/2}$.

(B1) Evaluate $\lim_{x\to 0} \dfrac{\cosh x - \cos x}{\sin x^2}$ in any way you wish.

(B2) Is the series $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{k^3 + \ln k + 1}$ absolutely convergent, conditionally convergent, or divergent? Justify your answer

(B3) Evaluate the integral $\displaystyle\int_{-1}^{1} \frac{2x}{1-x^2}\,dx$ or explain why you can't.

(B4) Give the Taylor series about 0 of the function $f(x) = \displaystyle\int_0^x \sin(t^2)\,dt$. For what values of x does the series converge to $f(x)$? Justify your answer.

(B5) Consider the sequence $\{a_k\}$ defined by $a_0 = 0$,

$$a_0 = 1, \quad a_{n+1} = 1 - a_n/2 \text{ for } n = 0, 1, 2, \ldots.$$

Show that the sequence converges to $2/3$.

(C1) Evaluate the limit $\lim_{x \to 0} \frac{\cosh x - \cos x}{x^2}$.

(C2) Evaluate the integral $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$ or explain why you can't.

(C3) Does the series $1 + \frac{1 \cdot 2}{1 \cdot 3} + \frac{1 \cdot 2 \cdot 3}{1 \cdot 3 \cdot 5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7} + \dots$ converge?

(C4) Test the following two series for (i) absolute and (ii) conditional convergence.

$$\text{(a)} \sum_{n=1}^{\infty} (-1)^k \frac{k+2}{k^2+k}$$

$$\text{(b)} \sum_{n=1}^{\infty} \frac{k^k}{k!}$$

(C5) Let S be the set of numbers of the form $(-1)^{n^2} \frac{n + 1/n!}{n+1}$ for $n \geq 2$. Explain why S does or does not have a least upper bound. If it has a least upper bound, what is it? Answer the same question about greatest lower bounds.

(D1) Evaluate $\lim_{x \to 0} \frac{\cos x - \cos 2x}{x \sin 4x}$.

(D2) Evaluate the integral $\int_0^1 \frac{dx}{x^{2/5}}$ or explain why you can't.

(D3) Use the formula $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$, valid for $-1 < x < 1$, to compute the Taylor series for $\tanh^{-1} x$. What is the interval of convergence for the series?

(D4) For each integer $n \geq 1$, let $a_n = 2\ln(3n-1) - \ln(2n^2 + 2n + 3)$. Does the sequence $\{a_n\}$ converge or diverge? If it converges, what is the limit?

(D5) (a) Does the series $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ converge or diverge?

(b) Does the series $\sum_{k=1}^{\infty} (-1)^k \frac{k+2}{k^3+k}$ converge absolutely, converge conditionally, or diverge?