Math 135: Homework 6

Due Thursday, February 9

1. Fix a real constant α. Show that $\displaystyle\sum_{k=0}^{\infty} \frac{\sin \alpha k}{2^k}$ converges. Evaluate the sum exactly (in terms of α) by using the fact that $\sin \alpha k = \mathrm{Im}(e^{i\alpha k})$. **Hints:** You may assume that all of the series we've been discussing work equally well with complex numbers as with real numbers. Also, note that for any real numbers a and b, we have $\frac{1}{a+ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$.

2. Fix an interval $I = (a, b)$. Let $y_1(x)$ and $y_2(x)$ be solutions of $y'' + p(x)y' + q(x)y = 0$ on I, where $p(x)$, $q(x)$ are continuous functions on I. Show that if there is a point in I where both y_1 and y_2 vanish or where both have maxima or minima, then one of y_1 and y_2 is a multiple of the other. [Hint: use existence and uniqueness of solutions, Theorem 19.2.]

3. In the following, you are given a differential equation and one solution of it. Use *reduction of order* to find the general solution. We will discuss this method in class on Monday or Tuesday: try a solution of the form $y_2(x) = u(x)y_1(x)$ and solve for the unknown function $u(x)$. When you plug $y_2(x)$ into the differential equation, the u terms will cancel, leaving an equation of the form $f(x)u'' + g(x)u' = 0$. This is a first order linear equation (and also separable) in the function u', so solve for u'. Integrate to get u, and then you can get the second solution $y_2 = uy_1$.

   (This technique is also discussed in the book: in the constant coefficient case, see Lesson 20C, and with nonconstant coefficients but using slightly different notation, see Lesson 23B.)

   (a) $x^2y'' - x(x+2)y' + (x+2)y = 0$, $y_1(x) = x$.

   (b) $xy'' - (x+2)y' + 2y = 0$, $y_1(x) = e^x$.