Math 135: Homework 2

Due Thursday, January 12

(1) Let $\{a_n\}$ be the sequence defined inductively by $a_1 = 1$, $a_{n+1} = \dfrac{1}{a_n^4 + 16}$.

(a) Show that $\{a_n\}$ is a Cauchy sequence.

(b) Show that $\{a_n\}$ converges to a solution of the equation $x^5 + 16x - 1 = 0$.

(c) Show that if $\{b_n\}$ is the sequence defined by $b_1 = 2$, $b_{n+1} = \dfrac{1}{b_n^4 + 16}$, then $\{b_n\}$ is convergent and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

Hint: Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1/(x^4 + 16)$.

(2) Show that the equation

$$x = 1 + \int_0^x \frac{\cos(t)}{t^2 + 4}\, dt$$

has one and only one solution.

(3) Recall that (by definition) $\lim_{x\to\infty} f(x) = L$ if and only if for every real number $\epsilon > 0$ there is a real number x_0 such that $|f(x) - L| < \epsilon$ for all $x > x_0$.

Prove the following:

$$\lim_{x\to\infty} f(x) = L \text{ if and only if } \lim_{t\to 0^+} f(1/t) = L\,.$$

(4) Show that for any real number c,

$$\lim_{x\to\infty} \left(\frac{x + c}{x - c}\right)^x = e^{2c}\,.$$