Sequences are often defined by recursion. For example, suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and x_0 is a real number. Then we may define a sequence $\{x_n\}$ iteratively

$$x_1 = f(x_0),\ x_2 = f(x_1),$$

and in general by the formula

$$x_{n+1} = f(x_n) \text{ for } n = 0, 1, 2, \ldots.$$

The purpose of this handout is to study this sequence, for a certain family of functions.

We begin with some definitions. Throughout this handout, $\Omega \subset \mathbb{R}$ denotes a set of one of the forms $[a, b]$ for $a < b$, $[a, \infty)$, $(-\infty, b]$, or $\mathbb{R}$; and f denotes a function of the form

$$f : \Omega \to \Omega$$

(i.e. Ω is the domain of f and the range of f is contained in Ω).

Definition 1. The function f is said to be a *contraction map* if there is a real number K with $0 < K < 1$ for which

$$|f(x) - f(y)| \leq K|x - y| \text{ for all } x, y \in \Omega.$$

For example, if $f(x)$ is differentiable and if there is a real number K with $0 < K < 1$ so that $|f'(x)| \leq K$ for all x in the interior of Ω, then f is a contraction. (Prove this!)

Lemma 2. *If f is a contraction map then f is continuous on Ω.*

Proof. Exercise. □

Definition 3. A point $x_0 \in \Omega$ is called a *fixed point* of f if $f(x_0) = x_0$.

Lemma 4. *A contraction map has at most one fixed point.*

Proof. Exercise □

Theorem 5. *Let $f : \Omega \to \Omega$ be a contraction map and let $x_0 \in \Omega$. Then the sequence $\{x_n\}$ defined inductively by $x_{n+1} = f(x_n)$, for $n \geq 0$, is a Cauchy sequence. Moreover, the limit $x_\infty = \lim_{n\to\infty} x_n$ is a fixed point of f.*

Corollary 6. *Let $f : \Omega \to \Omega$ be a contraction map. Then f has exactly one fixed point.*

We will prove the theorem through a series of lemmas.

Lemma 7. *Suppose $x_n \to x_\infty$. Then x_∞ is a fixed point.*

Proof. First note since Ω is either a closed interval or all of $\mathbb{R}$, then $x_\infty \in \Omega$. Hence $f(x_\infty)$ is defined. (Why?)

To see that $f(x_\infty) = x_\infty$, choose any $\epsilon > 0$. Then there is an integer $N > 0$ such that $|x_n - x_\infty| < \epsilon$ for all $n \geq N$. Choose any $n \geq N$, and use the triangle inequality to estimate as follows:

$$\begin{aligned} |f(x_\infty) - x_\infty| &= |f(x_\infty) - f(x_n) + f(x_n) - x_\infty| \leq |f(x_\infty) - f(x_n)| + |f(x_n) - x_\infty| \\ &\leq K|x_\infty - x_n| + |x_{n+1} - x_\infty| < 2\epsilon\,. \end{aligned}$$

Since ϵ was arbitrary, it follows that $f(x_\infty) = x_\infty$. □

Lemma 8. *The sequence $\{x_n\}$ is bounded. In particular, there is a real number R so that $|x_n - x_0| \leq R$ for all $n \geq 0$.*

Proof. Let $A = |x_1 - x_0|$. Observe that for any $n > 1$,

$$|x_n - x_{n-1}| = |f(x_{n-1}) - f(x_{n-2})| \leq K|x_{n-1} - x_{n-2}|\,.$$

Repeating this step n times yields the inequality

$$|x_n - x_{n-1}| \leq AK^{n-1},$$

valid for all $n \geq 1$. Thus,

$$\begin{aligned} |x_n - x_0| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots + (x_1 - x_0)| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_1 - x_0| \\ &\leq (K^{n-1} + K^{n-2} + \cdots + K + 1)A \leq \frac{A}{1-K}\,. \end{aligned}$$

Let $R = A/(1-K)$; then $|x_n - x_0| \leq R$ for all n. □

Lemma 9. *The sequence $\{x_n\}$ is Cauchy.*

Proof. Choose any $\epsilon > 0$. Since $0 < K < 1$, there is an integer $N > 0$ for which

$$2RK^N < \epsilon\,.$$

where R is as in the previous lemma.

We claim that $|x_n - x_m| < \epsilon$ for all $n, m \geq N$. To see this, note first that by definition of R,

$$|x_{n-N} - x_0| \leq R \text{ and } |x_{m-N} - x_0| \leq R\,.$$

Hence, by the triangle inequality, $|x_{n-N} - x_{m-N}| \leq 2R$. Now observe that

$$x_n = (\underbrace{f \circ f \circ \cdots \circ f}_{N})(x_{n-N}) \quad \text{and} \quad x_m = (\underbrace{f \circ f \circ \cdots \circ f}_{N})(x_{m-N})\,.$$

Therefore

$$|x_m - x_n| \leq K^N |x_{m-N} - x_{n-N}| \leq 2RK^N < \epsilon\,,$$

which is what we needed to prove. □

Example 10. Fix a number $c > 1$ and suppose that $\{x_n\}$ is a sequence defined inductively by $x_1 = 1$, $x_{n+1} = \sqrt{c + x_n}$. Then I claim that $\lim_{n \to \infty} x_n = \dfrac{1 + \sqrt{1 + 4c}}{2}$.

Consider the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{c + x}$. Then $f'(x) = \frac{1}{2\sqrt{c+x}}$. Since $c > 1$, for all $x > 0$ we have $|f'(x)| < 1/2$. Therefore f is a contraction, so it has a unique fixed point. By Theorem 5, the sequence $\{x_n\}$ converges to the fixed point. Furthermore, the fixed point is the number x_∞ so that $f(x_\infty) = x_\infty$:

$$\sqrt{c + x_\infty} = x_\infty.$$

Solving for x_∞ yields $x_\infty = \frac{1+\sqrt{1+4c}}{2}$.