Sequences are often defined by recursion. For example, suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a function and \( x_0 \) is a real number. Then we may define a sequence \( \{x_n\} \) iteratively

\[
x_1 = f(x_0), \quad x_2 = f(x_1),
\]
and in general by the formula

\[
x_{n+1} = f(x_n) \text{ for } n = 0, 1, 2, \ldots.
\]

The purpose of this handout is to study this sequence, for a certain family of functions.

We begin with some definitions. Throughout this handout, \( \Omega \subset \mathbb{R} \) denotes a set of one of the forms \([a, b]\) for \( a < b \), \((a, \infty]\), \((-\infty, b]\), or \(\mathbb{R} \); and \( f \) denotes a function of the form

\[
f : \Omega \to \Omega
\]
(i.e. \( \Omega \) is the domain of \( f \) and the range of \( f \) is contained in \( \Omega \)).

**Definition 1.** The function \( f \) is said to be a **contraction map** if there is a real number \( K \) with \( 0 < K < 1 \) for which

\[
|f(x) - f(y)| \leq K|x - y| \text{ for all } x, y \in \Omega.
\]

For example, if \( f(x) \) is differentiable and if there is a real number \( K \) with \( 0 < K < 1 \) so that \( |f'(x)| \leq K \) for all \( x \) in the interior of \( \Omega \), then \( f \) is a contraction. (Prove this!)

**Lemma 2.** If \( f \) is a contraction map then \( f \) is continuous on \( \Omega \).

**Proof.** Exercise.

**Definition 3.** A point \( x_0 \in \Omega \) is called a **fixed point** of \( f \) if \( f(x_0) = x_0 \).

**Lemma 4.** A contraction map has at most one fixed point.

**Proof.** Exercise

**Theorem 5.** Let \( f : \Omega \to \Omega \) be a contraction map and let \( x_0 \in \Omega \). Then the sequence \( \{x_n\} \) defined inductively by \( x_{n+1} = f(x_n) \), for \( n \geq 0 \), is a Cauchy sequence. Moreover, the limit \( x_\infty = \lim_{n \to \infty} x_n \) is a fixed point of \( f \).

**Corollary 6.** Let \( f : \Omega \to \Omega \) be a contraction map. Then \( f \) has exactly one fixed point.

We will prove the theorem through a series of lemmas.

**Lemma 7.** Suppose \( x_n \to x_\infty \). Then \( x_\infty \) is a fixed point.
Proof. First note since $\Omega$ is either a closed interval or all of $\mathbb{R}$, then $x_\infty \in \Omega$. Hence $f(x_\infty)$ is defined. (Why?)

To see that $f(x_\infty) = x_\infty$, choose any $\epsilon > 0$. Then there is an integer $N > 0$ such that $|x_n - x_\infty| < \epsilon$ for all $n \geq N$. Choose any $n \geq N$, and use the triangle inequality to estimate as follows:

$$|f(x_\infty) - x_\infty| = |f(x_\infty) - f(x_n) + f(x_n) - x_\infty| \leq |f(x_\infty) - f(x_n)| + |f(x_n) - x_\infty| \leq K|x_n - x_\infty| + |x_{n+1} - x_\infty| < 2\epsilon.$$ 

Since $\epsilon$ was arbitrary, it follows that $f(x_\infty) = x_\infty$. \hfill \Box

**Lemma 8.** The sequence $\{x_n\}$ is bounded. In particular, there is a real number $R$ so that $|x_n - x_0| \leq R$ for all $n \geq 0$.

**Proof.** Let $A = |x_1 - x_0|$. Observe that for any $n > 1$,

$$|x_n - x_{n-1}| = |f(x_{n-1}) - f(x_{n-2})| \leq K|x_{n-1} - x_{n-2}|.$$ 

Repeating this step $n$ times yields the inequality

$$|x_n - x_{n-1}| \leq AK^{n-1},$$

valid for all $n \geq 1$. Thus,

$$|x_n - x_0| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots + (x_1 - x_0)| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_1 - x_0| \leq (K^{n-1} + K^{n-2} + \cdots + K + 1)A \leq \frac{A}{1 - K}.$$

Let $R = A/(1 - K)$; then $|x_n - x_0| \leq R$ for all $n$. \hfill \Box

**Lemma 9.** The sequence $\{x_n\}$ is Cauchy.

**Proof.** Choose any $\epsilon > 0$. Since $0 < K < 1$, there is an integer $N > 0$ for which

$$2RK^N < \epsilon.$$ 

where $R$ is as in the previous lemma.

We claim that $|x_n - x_m| < \epsilon$ for all $n, m \geq N$. To see this, note first that by definition of $R$,

$$|x_{n-N} - x_0| \leq R \text{ and } |x_{m-N} - x_0| \leq R.$$ 

Hence, by the triangle inequality, $|x_{n-N} - x_{m-N}| \leq 2R$. Now observe that

$$x_n = (f \circ f \circ \cdots \circ f)(x_{n-N}) \text{ and } x_m = (f \circ f \circ \cdots \circ f)(x_{m-N}).$$

Therefore

$$|x_m - x_n| \leq K^N|x_{m-N} - x_{n-N}| \leq 2RK^N < \epsilon,$$

which is what we needed to prove. \hfill \Box
Example 10. Fix a number $c > 1$ and suppose that $\{x_n\}$ is a sequence defined inductively by $x_1 = 1$, $x_{n+1} = \sqrt{c + x_n}$. Then I claim that $\lim_{n \to \infty} x_n = \frac{1 + \sqrt{1 + 4c}}{2}$.

Consider the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{c + x}$. Then $f'(x) = \frac{1}{2\sqrt{c + x}}$. Since $c > 1$, for all $x > 0$ we have $|f'(x)| < 1/2$. Therefore $f$ is a contraction, so it has a unique fixed point. By Theorem 5, the sequence $\{x_n\}$ converges to the fixed point. Furthermore, the fixed point is the number $x_\infty$ so that $f(x_\infty) = x_\infty$:

$$\sqrt{c + x_\infty} = x_\infty.$$  

Solving for $x_\infty$ yields $x_\infty = \frac{1 + \sqrt{1 + 4c}}{2}$.  