1 Preliminaries

If $f(t)$ is defined on the interval $[0, \infty)$, then its Laplace transform is defined to be

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) \, dt,$$

as long as this integral is defined and converges. In particular, if $f$ is of exponential order and is piecewise continuous, the Laplace transform of $f(t)$ will be defined.

- $f$ is of exponential order if there are constants $M$ and $c$ so that

$$|f(t)| \leq Me^{ct}.$$

Since the integral $\int_0^\infty e^{-st}Me^{ct} \, dt$ converges if $s > c$, then by a comparison test (like (11.7.2) in Salas-Hille-Etgen), the integral defining the Laplace transform of $f(t)$ will converge.

- $f$ is piecewise continuous if over each interval $[0, b]$, $f(t)$ has only finitely many discontinuities, and at each point $a$ in $[0, b]$, both of the limits

$$\lim_{t \to a^-} f(t) \quad \text{and} \quad \lim_{t \to a^+} f(t)$$

exist – they need not be equal, but they must exist. (At the endpoints 0 and $b$, the appropriate one-sided limits must exist.)

2 Step functions

Define $u(t)$ to be the function

$$u(t) = \begin{cases} 
0 & \text{if } t < 0, \\
1 & \text{if } t \geq 0.
\end{cases}$$

Then $u(t)$ is called the step function, or sometimes the Heaviside step function: it jumps from 0 to 1 at $t = 0$. Note that for any number $a > 0$, the graph of the function $u(t - a)$ is the same as the graph of $u(t)$, but translated right by $a$: $u(t - a)$ jumps from 0 to 1 at $t = a$. 

![Graph of step functions](image-url)
Example 1. We can use the step function to write formulas for functions which are defined piecewise: if \( g(t) \) is defined as

\[
g(t) = \begin{cases} L(t) & \text{if } t < a, \\ R(t) & \text{if } t \geq a, \end{cases}
\]

then

\[
g(t) = L(t) + [R(t) - L(t)] u(t - a).
\]

Why? If \( t < a \), then the step function \( u(t - a) \) is zero, so this turns into \( L(t) \). If \( t \geq a \), then the step function equals 1, so the \( L(t) \) terms cancel and we are left with \( R(t) \). We can get similar expressions for functions defined in three or more separate pieces.

Example 2. Express the function

\[
g(t) = \begin{cases} t & \text{if } t < 1, \\ t^2 & \text{if } 1 \leq t < 3, \\ t^3 & \text{if } t \geq 3 \end{cases}
\]

using the step function.

We will work from left to right on the real line. When \( t < 1 \), the expression \( t \) is correct, but it’s not right for \( t \geq 1 \). Between 1 and 3, we have to subtract \( t \) and add \( t^2 \). We can do this by adding an appropriate function multiplied by \( u(t - 1) \), and in particular, this expression is correct when \( t < 3 \):

\[
t + u(t - 1) \left[t^2 - t\right].
\]

Finally, when \( t \geq 3 \), we need to subtract \( t^2 \) and add \( t^3 \), so we get this formula for \( g(t) \):

\[
g(t) = t + u(t - 1) \left[t^2 - t\right] + u(t - 3) \left[t^3 - t^2\right].
\]

Proposition 3. The Laplace transform of \( u(t - a) \) is \( e^{-as}/s \). If \( f(t) \) is a function with Laplace transform \( F(s) \), then

\[
L \left(u(t - a)f(t - a)\right) = e^{-as}F(s).
\]

Proof. The integral defining the Laplace transform is

\[
L \left(u(t - a)f(t - a)\right) = \int_0^\infty e^{-st}u(t - a)f(t - a) \, dt = \int_a^\infty e^{-st}f(t - a) \, dt.
\]

(The second equality is by the definition of the step function.) Now make a change of variables: let \( w = t - a \). When \( t = a, w = 0 \), and when \( t = \infty, w = \infty \), so the integral becomes

\[
\int_0^\infty e^{-s(w+a)}f(w) \, dw = \int_0^\infty e^{-sw}e^{-sa}f(w) \, dw = e^{-sa}\int_0^\infty e^{-sw}f(w) \, dw = e^{-sa}L \left(f(t)\right).
\]

Example 4. \( L \left(u(t - 3)(t - 3)^2\right) = e^{-3s}\frac{2!}{s^3} \).
By the way, since the Laplace transform is defined in terms of an integral, the behavior at the discontinuities of piecewise-defined functions is not important. For example, the following functions will have the same Laplace transform:

\[
g(t) = \begin{cases} 0 & \text{if } t < 1, \\ t & \text{if } t \geq 1, \end{cases} \\
h(t) = \begin{cases} 0 & \text{if } t \leq 1, \\ t & \text{if } t > 1. \end{cases}
\]

**Exercises 5.**

(a) Suppose \( f(t) \) is some function. In terms of the graph of \( f(t) \), what does the function \( u(t-a)f(t-a) \) look like?

(b) Compute \( \mathcal{L}(u(t-2)) \).

(c) Compute \( \mathcal{L}(u(t-2)\sin(t-2)) \).

**Example 6.** What is the Laplace transform of the function \( g(t) \) defined by

\[
g(t) = \begin{cases} 0 & \text{when } t < 1, \\ t-1 & \text{when } t \geq 1? \end{cases}
\]

To answer this, we need to write \( g(t) \) in terms of the step function, but that’s easy here: \( g(t) = u(t-1)(t-1) \). So in the notation of Proposition 3, we could write this as \( g(t) = u(t-1)f(t-1) \), where \( f(t) = t \). So by the proposition,

\[
\mathcal{L}(g(t)) = e^{-s} \mathcal{L}(t) = e^{-s} \frac{1}{s^2}.
\]

**Example 7.** Suppose that \( g(t) \) is the function defined by

\[
g(t) = \begin{cases} t & \text{if } t < 1, \\ t^2 & \text{if } 1 \leq t < 3, \\ t^3 & \text{if } t \geq 3 \end{cases}
\]

What is \( \mathcal{L}(g(t)) \)?

In Example 2 we found an expression for \( g(t) \) in terms of step functions:

\[
g(t) = t + u(t-1)(t^2 - t) + u(t-3)(t^3 - t^2).
\]

Unfortunately, this isn’t in the right form to apply the proposition, because the terms don’t look like \( u(t-a)f(t-a) \) for some function \( f \). Let’s look at the summands: the first summand is just \( t \), and we can compute its Laplace transform: it’s just \( 1/s^2 \). The second summand is \( u(t-1)(t^2 - t) \), and so we need to write \( t^2 - t \) as a function of \( t-1 \): we want a function \( f \) so that \( f(t-1) = t^2 - t \). If we make the substitution \( x = t-1 \), then \( t = x+1 \), and this becomes \( f(x) = (x+1)^2 - (x+1) = x^2 + x \). This is our formula for \( f \): \( f(x) = x^2 + x \), and so \( f(t) = t^2 + t \).

Similarly, the third summand is \( u(t-3)(t^3 - t^2) \), and so we want to write \( t^3 - t^2 \) as \( f_2(t-3) \) for some function \( f_2 \). We want \( f_2(t-3) = t^3 - t^2 \), so letting \( x = t-3 \), we get \( f_2(x) = (x+3)^3 - (x+3)^2 = x^3 + 8x^2 + 21x + 18 \).
Now we can compute the Laplace transform of $g(t)$:

$$
\mathcal{L}(g(t)) = \mathcal{L}(t + u(t - 1)f(t - 1) + u(t - 3)f_2(t - 3)) \\
= \mathcal{L}(t) + \mathcal{L}(u(t - 1)f(t - 1)) + \mathcal{L}(u(t - 3)f_2(t - 3)) \\
= \frac{1}{s^2} + e^{-s}\mathcal{L}(f(t)) + e^{-3s}\mathcal{L}(f_2(t)) \\
= \frac{1}{s^2} + e^{-s}(t^2 + t) + e^{-3s}(t^3 + 8t^2 + 21t + 18) \\
= \frac{1}{s^2} + e^{-s}\left(\frac{2}{s^3} + \frac{1}{s^2}\right) + e^{-3s}\left(\frac{3!}{s^4} + \frac{8 \cdot 2!}{s^3} + \frac{21}{s^2} + \frac{18}{s}\right).
$$

**Example 8.** Suppose that $g(t)$ is defined by

$$
g(t) = \begin{cases} 
  t & \text{when } 0 \leq t < 2, \\
  0 & \text{when } t \geq 2.
\end{cases}
$$

Solve the initial value problem

$$
y'' + 4y = g(t), \quad y(0) = 0, \; y'(0) = 0.
$$

I would like to use Laplace transforms, and to compute the Laplace transform of $g(t)$, I need to write $g(t)$ using step functions. When computing the Laplace transform of a function $g(t)$, what happens when $t < 0$ is irrelevant – the integral starts at $t = 0$ – so I can write $g(t)$ like this:

$$
g(t) = t + u(t - 2)(-t) = t + u(t - 2)(-(t - 2) - 2).
$$

So if $f(t) = -t - 2$, then $g(t) = t + u(t - 2)f(t - 2)$. Now we’re ready to go: let $Y = \mathcal{L}(y)$, and then because of the initial conditions, $\mathcal{L}(y') = sY$ and $\mathcal{L}(y'') = s^2Y$. When I apply the Laplace transform to the differential equation, I get

$$
(s^2 + 4)Y = \mathcal{L}(t + u(t - 2)f(t - 2)) = \frac{1}{s^2} + e^{-2s}\left(-\frac{1}{s^2} - \frac{2}{s}\right).
$$

Therefore

$$
Y = (1 - e^{-2s})\left(\frac{1}{s^2} + \frac{1}{s} + \frac{1}{s^2 + 4}\right) - 2e^{-2s}\left(-\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s^2 + 4}\right) \\
= \left(1 - e^{-2s}\right)\left(\frac{1/4}{s^2} + \frac{-1/4}{s^2 + 4}\right) - 2e^{-2s}\left(\frac{1/4}{s} + \frac{-s/4}{s^2 + 4}\right) \\
= \left(\frac{1/4}{s^2} + \frac{-1/4}{s^2 + 4}\right) + e^{-2s}\left(-\frac{1/4}{s^2} + \frac{1/4}{s^2 + 4} - \frac{1/2}{s} + \frac{s/2}{s^2 + 4}\right).
$$
Therefore
\[
y = \mathcal{L}^{-1} \left( \frac{1}{4} + \frac{1}{8} \sin 2t + u(t) f(t - 2) \right),
\]
where
\[
f(t) = \mathcal{L}^{-1} \left( \frac{1}{4} + \frac{1}{8} \sin 2t - \frac{1}{2} + \frac{1}{2} \cos 2t \right).
\]

So
\[
y(t) = \frac{1}{4} t - \frac{1}{8} \sin 2t + u(t - 2) \left( \frac{1}{4} t - \frac{1}{8} \sin 2(t - 2) - \frac{1}{2} + \frac{1}{2} \cos 2(t - 2) \right) + \frac{1}{4} t - \frac{1}{8} \sin 2t + u(t - 2) \left( \frac{1}{4} t + \frac{1}{8} \sin 2(t - 2) + \frac{1}{2} \cos 2(t - 2) \right).
\]

We can also write this as a piecewise-defined function:
\[
y(t) = \begin{cases} 
\frac{1}{4} t - \frac{1}{8} \sin 2t & \text{if } t < 2, \\
-\frac{1}{8} \sin 2t + \frac{1}{8} \sin 2(t - 2) + \frac{1}{2} \cos 2(t - 2) & \text{if } t \geq 2.
\end{cases}
\]

You can check that this function is continuous and twice differentiable for all \( t \geq 0 \). (The only interesting point is \( t = 2 \).) Here’s a graph:

**Example 9.** Suppose that \( g(t) \) is defined by
\[
g(t) = \begin{cases} 
100 \sin 40t & \text{when } 0 \leq t < 7, \\
0 & \text{when } t \geq 7.
\end{cases}
\]
Solve the initial value problem

\[ y'' + 3y' + 2y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \]

I’m going to skip the details and just reproduce the solution and its graph; we’ll discuss some interpretations of this example in class:

\[
y(t) = \begin{cases} 
-2.49e^{-2t} + 2.50e^{-t} - 0.0622\sin(40t) - 0.00467\cos(40t), & \text{if } 0 \leq t < 7, \\
-2.25e^{-2(t-7)} + 2.28e^{-(t-7)}, & \text{if } t \geq 7.
\end{cases}
\]

**Exercises 10.** Solve the initial value problems

\[ y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0, \]

where \( f(t) \) is as given in each of the following pictures.
3 The Dirac $\delta$-function

Proposition 11. (a) Let $\epsilon$ be a positive number and consider the function $f_\epsilon(t)$ defined by

$$f_\epsilon(t) = \begin{cases} 
1/\epsilon & \text{if } 0 \leq t \leq \epsilon, \\
0 & \text{if } t > \epsilon.
\end{cases}$$

Then

$$\mathcal{L}(f_\epsilon(t)) = \frac{1 - e^{s\epsilon}}{s\epsilon}.$$ 

(b) “Define” the Dirac delta function $\delta(t)$ to be

$$\delta(t) = \lim_{\epsilon \to 0^+} f_\epsilon(t).$$

Then $\delta(t) = 0$ except when $t = 0$, and it has the following properties with respect to integration: for any function $f(t)$,

$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1, \quad \int_{-\infty}^{\infty} \delta(t) f(t) \, dt = f(0).$$

Therefore for any positive number $a$, we have $\mathcal{L}(\delta(t - a)) = e^{-as}$.

The Dirac delta function models an instantaneous force applied to a system, like hitting a mass with a hammer.

Now, $\delta(t)$ is not actually a function: the limit defining it doesn’t exist when $t = 0$, for one thing. If there were a way to define it, then properties of integrals show that if $g(t)$ is any function with $g(t) = 0$ whenever $t \neq 0$, then $\int_{a}^{b} g(t) = 0$ for any $a$ and $b$. Instead, $\delta(t)$ is what is called a generalized function or distribution, and although it isn’t a function, it can be treated like one in many ways. Really its defining property is that for any function $f(t)$,

$$\int_{-\infty}^{\infty} \delta(t) f(t) \, dt = f(0).$$

Example 12. Solve the initial value problem

$$y'' + 2y' + 2y = \delta(t - 1), \quad y(0) = 0, \quad y'(0) = 0.$$ 

We apply the Laplace transform. If $Y = \mathcal{L}(y)$, then $\mathcal{L}(y') = sY$ and $\mathcal{L}(y'') = s^2Y$, and the equation becomes

$$(s^2 + 2s + 2)Y = e^{-s}, \quad \text{so} \quad Y = e^{-s} \frac{1}{s^2 + 2s + 2}.$$ 

We complete the square and write this as

$$Y = e^{-s} \frac{1}{s^2 + 2s + 2} = e^{-s} \frac{1}{(s + 1)^2 + 1}.$$
Therefore
\[ y = \mathcal{L}^{-1}(Y) = u(t-1)f(t-1), \]
where
\[ f(t) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2 + 1}\right) = e^{-t}\sin t. \]
So
\[ y(t) = u(t-1)e^{-(t-1)}\sin(t-1), \]
or
\[ y(t) = \begin{cases} 
0 & \text{if } t < 1, \\
e^{-(t-1)}\sin(t-1) & \text{if } t \geq 1.
\end{cases} \]

Note that this function is continuous everywhere, but it is not differentiable at \( t = 1 \). This is not surprising, because \( t = 1 \) is when the delta function is applied – this example models what happens in a damped spring system when you hit the mass with a hammer.

**Exercises 13.**
(a) Solve the initial value problem
\[ y'' + 3y' + 2y = \delta(t-1), \quad y(0) = 0, \quad y'(0) = 0. \]
(b) Consider the initial value problem
\[ y'' + 4y = \delta(t) + c\delta(t-\pi), \quad y(0) = 0, \quad y'(0) = 0, \]
where \( c \) is a constant. What should \( c \) be so that the solution stops completely at time \( \pi \)? That is, what should \( c \) be so that the solution has the form
\[ y = \begin{cases} 
?? & \text{if } 0 \leq t < \pi, \\
0 & \text{if } t \geq \pi.
\end{cases} \]