The *convolution* of functions $f(t)$ and $g(t)$ is defined to be

$$(f * g)(t) = \int_0^t f(t-u)g(u)\,du.$$

(The asterisk $*$ does not mean ordinary multiplication: it is a new operation, convolution, defined by the integral on the right side.)

Properties of convolution:

- $f * g = g * f$
- $(f * g) * h = f * (g * h)$
- $f * 0 = 0 = 0 * f$ (but note that $f * 1 \neq f$)
- $\delta * f = f = f * \delta$
- and most importantly for our purposes, Theorem 27.7 in Tenenbaum-Pollard says that if $\mathcal{L}\left(f(t)\right) = F(s)$ and $\mathcal{L}\left(g(t)\right) = G(s)$, then

$$\mathcal{L}\left((f * g)(t)\right) = F(s)G(s).$$

1. Compute the convolution of e^{-2t} and e^{-3t}.
2. Compute the convolution of e^{at} and e^{bt} for any nonzero constants a and b.
3. Compute the convolution of e^{-t} and 1.
4. Compute the convolution of e^{-t} and t.
5. Compute the convolution of e^{-t} and t^2.
6. Compute the convolution of e^{-t} and $\cos t$.
7. Compute the convolution of e^{-t} and $\sin t$.
8. Compute the convolution of $\cos 2t$ and $\sin t$.
9. Compute the convolution of $\cos 2t$ and $\sin 2t$.

In class we will discuss this method for solving an initial value problem: given

$$ay'' + by' + cy = g(t), \quad y(0) = y_0,\ y'(0) = y_1,$$

let $e(t)$ be the *unit impulse response function*: $e(t)$ is the solution to

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0,\ y'(0) = 0.$$

Then $e(t) = \mathcal{L}^{-1}\left(\frac{1}{as^2+bs+c}\right)$.

For the original initial value problem, the *state-free solution* y_s is the solution to

$$ay'' + by' + cy = g(t), \quad y(0) = 0,\ y'(0) = 0.$$

(That is, the same equation but with zero initial conditions.) The *input-free solution* y_i is the solution to

$$ay'' + by' + cy = 0, \quad y(0) = y_0,\ y'(0) = y_1.$$

(That is, the associated homogeneous equation but with the same initial conditions.) Then the state-free solution is the convolution

$$y_s = e(t) * g(t),$$

and the input-free solution is

$$y_i = ay_0 e'(t) + (ay_1 + by_0)e(t).$$

The function $y_s + y_i$ is the solution to the original initial value problem.

Use this to solve these initial value problems:

10. $y'' + y = 2$, $y(0) = 0$, $y'(0) = 1$
11. $y'' + y = e^t$, $y(0) = 0$, $y'(0) = 0$
12. $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$ (answer will involve $g(t)$)
13. $y'' - 5y' + 4y = g(t)$, $y(0) = 1$, $y'(0) = -1$
14. $y'' + 4y' + 3y = g(t)$, $y(0) = 0$, $y'(0) = 0$
15. $y'' + 2y' + 2y = g(t)$, $y(0) = 1$, $y'(0) = -2$