The point of this handout is Theorem 2: a way of proving that a sequence converges even if you can't tell what the limit is.

Definition 1. A sequence $\{a_n\}$ is said to be *Cauchy* (or to be a *Cauchy sequence*) if for every real number $\epsilon > 0$, there is an integer N (possibly depending on ϵ) for which

$$|a_n - a_m| < \epsilon \text{ for all } n, m \ge N.$$
(1)

Theorem 2. A sequence of real numbers is convergent if and only if it is Cauchy.

Proof. (\Rightarrow) Let $\{a_n\}$ be a convergent sequence with limit L. To verify that $\{a_n\}$ is Cauchy, begin by choosing a number $\epsilon > 0$. We must show that there is an integer N for which (1) holds.

But since a_n converges to L, there is an integer N > 0 for which $|a_n - L| < \epsilon/2$ for all $n \ge N$. Notice that for all n, m > N we may estimate as follows:

$$|a_n - a_m| = |(a_n - L) - (a_m - L)|$$

$$\leq |a_n - L| + |a_m - L| \quad \text{(by the triangle inequality)}$$

$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $\{a_n\}$ is Cauchy.

 (\Leftarrow) Now let $\{a_n\}$ be a Cauchy sequence. We want to show that $\{a_n\}$ converges.

First notice that $\{a_n\}$ is bounded. To see this, let $\epsilon = 1$. Then there is an integer N such that $|a_n - a_m| < 1$ for all n, m > N. Set m = N + 1; then for all n > N, we have

$$(a_{N+1}) - 1 < a_n < (a_{N+1}) + 1.$$

Let $U = \max\{a_1, ..., a_N, a_{N+1} + 1\}$ and $L = \min\{a_1, ..., a_N, a_{N+1} - 1\}$. Clearly, $L \le a_n \le U$ for all n, so $\{a_n\}$ is bounded.

Now let $\{b_n\}$ and $\{c_n\}$ be the bounded sequences defined by

$$b_n = \underset{m \ge n}{\operatorname{glb}} a_m$$
 and $c_n = \underset{m \ge n}{\operatorname{lub}} a_m$,

and notice that by construction the following inequalities are satisfied:

$$b_n \le a_m \le c_n \text{ for all } m \ge n.$$
 (2)

Finally notice that $\{b_n\}$ is nondecreasing and $\{c_n\}$ is nonincreasing (see Exercise 29, page 532). Hence, by Theorem 11.3.6, both $\{b_n\}$ and $\{c_n\}$ converge. Let $B = \lim_{n \to \infty} b_n$ and $C = \lim_{n \to \infty} c_n$. By the theorem, the following inequalities hold:

$$b_n \leq B \leq C \leq c_n$$
 for all n .

We will show that $\{a_n\}$ converges by showing B = C and applying the pinching lemma.

Begin by choosing any $\epsilon > 0$. Then there is an integer N for which

$$|a_n - a_m| < \epsilon$$
 for all $n, m \ge N$.

In particular,

$$a_N - \epsilon < a_m$$
 for all $m \ge N$,

showing that $a_N - \epsilon \leq b_N \leq B$. Similarly

$$a_m < a_N + \epsilon$$
 for all $m \ge N$.

showing that $C \leq c_N \leq a_N + \epsilon$. Hence,

$$a_N - \epsilon < B \le C < a_N + \epsilon \,.$$

It follows that $C - B < 2\epsilon$ for every $\epsilon > 0$, which implies that B = C.

Since $b_n \leq a_n \leq c_n$ and $\lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = B$, the pinching lemma applies to show that $a_n \to B$, concluding the proof.