The point of this handout is Theorem 2: a way of proving that a sequence converges even if you can't tell what the limit is.

Definition 1. A sequence $\left\{a_{n}\right\}$ is said to be Cauchy (or to be a Cauchy sequence) if for every real number $\epsilon>0$, there is an integer $N$ (possibly depending on $\epsilon$ ) for which

$$
\begin{equation*}
\left|a_{n}-a_{m}\right|<\epsilon \text { for all } n, m \geq N . \tag{1}
\end{equation*}
$$

Theorem 2. A sequence of real numbers is convergent if and only if it is Cauchy.
Proof. $(\Rightarrow)$ Let $\left\{a_{n}\right\}$ be a convergent sequence with limit $L$. To verify that $\left\{a_{n}\right\}$ is Cauchy, begin by choosing a number $\epsilon>0$. We must show that there is an integer $N$ for which (1) holds.
But since $a_{n}$ converges to $L$, there is an integer $N>0$ for which $\left|a_{n}-L\right|<\epsilon / 2$ for all $n \geq N$. Notice that for all $n, m>N$ we may estimate as follows:

$$
\begin{aligned}
\left|a_{n}-a_{m}\right| & =\left|\left(a_{n}-L\right)-\left(a_{m}-L\right)\right| \\
& \leq\left|a_{n}-L\right|+\left|a_{m}-L\right| \quad \text { (by the triangle inequality) } \\
& <\epsilon / 2+\epsilon / 2=\epsilon .
\end{aligned}
$$

Thus $\left\{a_{n}\right\}$ is Cauchy.
$(\Leftarrow)$ Now let $\left\{a_{n}\right\}$ be a Cauchy sequence. We want to show that $\left\{a_{n}\right\}$ converges.
First notice that $\left\{a_{n}\right\}$ is bounded. To see this, let $\epsilon=1$. Then there is an integer $N$ such that $\left|a_{n}-a_{m}\right|<1$ for all $n, m>N$. Set $m=N+1$; then for all $n>N$, we have

$$
\left(a_{N+1}\right)-1<a_{n}<\left(a_{N+1}\right)+1 .
$$

Let $U=\max \left\{a_{1}, \ldots, a_{N}, a_{N+1}+1\right\}$ and $L=\min \left\{a_{1}, \ldots, a_{N}, a_{N+1}-1\right\}$. Clearly, $L \leq a_{n} \leq U$ for all $n$, so $\left\{a_{n}\right\}$ is bounded.
Now let $\left\{b_{n}\right\}$ and $\left\{c_{n}\right\}$ be the bounded sequences defined by

$$
b_{n}=\operatorname{glb}_{m \geq n}^{\operatorname{ll}} a_{m} \quad \text { and } \quad c_{n}=\operatorname{lub}_{m \geq n} a_{m},
$$

and notice that by construction the following inequalities are satisfied:

$$
\begin{equation*}
b_{n} \leq a_{m} \leq c_{n} \text { for all } m \geq n \tag{2}
\end{equation*}
$$

Finally notice that $\left\{b_{n}\right\}$ is nondecreasing and $\left\{c_{n}\right\}$ is nonincreasing (see Exercise 29, page 532). Hence, by Theorem 11.3.6, both $\left\{b_{n}\right\}$ and $\left\{c_{n}\right\}$ converge. Let $B=\lim _{n \rightarrow \infty} b_{n}$ and $C=\lim _{n \rightarrow \infty} c_{n}$. By the theorem, the following inequalities hold:

$$
b_{n} \leq B \leq C \leq c_{n} \text { for all } n .
$$

We will show that $\left\{a_{n}\right\}$ converges by showing $B=C$ and applying the pinching lemma.

Begin by choosing any $\epsilon>0$. Then there is an integer $N$ for which

$$
\left|a_{n}-a_{m}\right|<\epsilon \text { for all } n, m \geq N .
$$

In particular,

$$
a_{N}-\epsilon<a_{m} \text { for all } m \geq N,
$$

showing that $a_{N}-\epsilon \leq b_{N} \leq B$. Similarly

$$
a_{m}<a_{N}+\epsilon \text { for all } m \geq N,
$$

showing that $C \leq c_{N} \leq a_{N}+\epsilon$. Hence,

$$
a_{N}-\epsilon<B \leq C<a_{N}+\epsilon .
$$

It follows that $C-B<2 \epsilon$ for every $\epsilon>0$, which implies that $B=C$.
Since $b_{n} \leq a_{n} \leq c_{n}$ and $\lim _{n \rightarrow \infty} b_{n}=\lim _{n \rightarrow \infty} c_{n}=B$, the pinching lemma applies to show that $a_{n} \rightarrow B$, concluding the proof.

