Suppose that f is a positive nonincreasing function on the interval $[m, \infty)$. Then the integral test applies to the series $\displaystyle\sum_{k=m}^{\infty} f(k)$; however, the proof of the integral test yields more information than the test itself. While the test just says whether the series converges or diverges, the proof allows us to estimate the value of the series (if it converges) or of its partial sums (in general).

The crucial point is the pair of inequalities

$$f(m+1) + \cdots + f(n) \le \int_m^n f(x)\,dx \le f(m) + \cdots + f(n-1).$$

(See the last line on p. 586 in Salas-Hille-Etgen for the $m = 1$ case; the same argument works in general.) These inequalities imply that

$$\int_m^{n+1} f(x)\,dx \le f(m) + \cdots + f(n) \le f(m) + \int_m^n f(x)\,dx, \tag{1}$$

and these can be used to estimate the partial sums of the series $\sum f(k)$, whether the series converges or not. If it does converge, letting $n \to \infty$ turns (1) into

$$\int_m^{\infty} f(x)\,dx \le \sum_{k=m}^{\infty} f(k) + \cdots + f(n) \le f(m) + \int_m^{\infty} f(x)\,dx. \tag{2}$$

You can estimate the sum of the whole series by taking $m = 1$ here. For better accuracy, you can add up the first few terms by hand and use (2) to estimate the remainder.

Example 3. Let's estimate $\sum_{k=1}^{\infty} 1/k^2$. Since $\int_m^{\infty} x^{-2}\,dx = -x^{-1}\big|_m^{\infty} = m^{-1}$, (2) tells us (with $m = 1$) that

$$1 \le \sum_{k=1}^{\infty} \frac{1}{k^2} \le 1 + 1.$$

We can do better by computing the sum of the first nine terms by machine:

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{81} = 1.5397\ldots.$$

Then apply (2) with $m = 10$ to get

$$0.1 = \int_{10}^{\infty} x^{-2}\,dx \le \sum_{k=10}^{\infty} \frac{1}{k^2} \le \frac{1}{10^2} + \int_{k=10}^{\infty} x^{-2}\,dx = 0.11.$$

So we can add the first 9 terms back to get

$$1.6397\cdots \le \sum_{k=1}^{\infty} \frac{1}{k^2} \le 1.6497\ldots.$$

This means that to two decimal places, $\sum 1/k^2 \approx 1.64$. (The actual sum is $\pi^2/6 \approx 1.6449341$.)

Example 4. Let's estimate the partial sums of the harmonic series $\sum_{k=1}^{\infty} 1/k$. For any n, (1) tells us that

$$\int_1^{n+1} \frac{dx}{x} \leq \sum_{k=1}^{n} \frac{1}{k} \leq 1 + \int_1^n \frac{dx}{x},$$

which is to say,

$$\ln(n+1) \leq \sum_{k=1}^{n} \frac{1}{k} \leq 1 + \ln n.$$

We know that $\sum_{k=1}^{\infty} 1/k$ diverges. How big must n be for the partial sum $\sum_{k=1}^{n} 1/k$ to be larger than 1000? We solve the following inequality for n:

$$1000 < 1 + \ln n, \quad \text{or} \quad e^{999} < n.$$

So the nth partial sum will be greater than 1000 if $n > e^{999}$, which is approximately 7.25×10^{433}.