Know the formal definitions of scalar product, positive definite, orthonormal basis, characteristic polynomial, eigenvalue, eigenvectors.

Know how to compute eigenvalues, eigenvectors, and bases of eigenvectors (when they exist).

Do the following problems in Lang:

> p.178#2, p.178#3, p.179#13, p.189#4, p.189#7, p.208#5(a),
> p.217#1(b) (by Cramer's rule and by Gauss elimination), p.221#2, p.237#1, p.237#4,
> p.249#2–8, p.249#15, p.260#6

Also do the following:

(1) Let $A = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. Find a matrix B such that $B^2 = A$. **Hint:** *First diagonalize* A.

(2) Solve the initial value problem

$$\frac{dx_1}{dt} + 2x_1 - x_2 = 0 \qquad\qquad x_1(0) = 6$$
$$\frac{dx_2}{dt} - x_1 + 2x_2 = 0 \qquad\qquad x_2(0) = 7$$

as follows:

(i) Write the problem in vector form:

$$\frac{d}{dt}X + AX = O \qquad\qquad X(0) = \begin{pmatrix} 6 \\ 7 \end{pmatrix}.$$

(ii) Find an orthonormal basis of eigenvectors of the matrix A.

(iii) Find two independent solutions $X = X_1(t)$ and $X = X_2(t)$ of $dX/dt + AX = O$ with

$$X_1(t) = f_1(t)B_1 \text{ and } X_2(t) = f_2(t)B_2\,,$$

where $\{B_1, B_2\}$ is an orthonormal basis of $\mathbb{R}^2$, and f_1 and f_2 are real valued functions.

(iv) Using (ii), write the general solution in the form

$$X(t) = B \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where the columns of B are B_1 and B_2.

(v) Show that $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B^t \begin{pmatrix} 6 \\ 7 \end{pmatrix}$.

(vi) Put this all together to write the solution in the form

$$X(t) = B \begin{pmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{pmatrix} B^t X_0$$

(3) Let $A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ be a non-zero 3×3 skew-symmetric matrix.

(i) Show that the characteristic polynomial of A is of the form

$$p_A(t) = t^3 + (a^2 + b^2 + c^2)t = t(t - \lambda_1)(t - \lambda_2) = t(t^2 - (\lambda_1 + \lambda_2)t + \lambda_1\lambda_2)\,,$$

and from this conclude that the spectrum of A (that is, the list of eigenvalues of A) is of the form $\{0, \lambda_1, \lambda_2\}$, with

$$\lambda_1 + \lambda_2 = 0 \quad \text{and} \quad \lambda_1\lambda_2 = a^2 + b^2 + c^2\,.$$

Conclude that $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$, with $\omega = \sqrt{a^2 + b^2 + c^2}$

(ii) Let $B_1 = \frac{1}{\omega} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Show that $AB_1 = O$.

(iii) Let $B_2 + iB_3$ satisfy

$$A(B_2 + iB_3) = i\omega(B_2 + iB_3), \quad \text{so} \quad AB_2 = -\omega\, B_3 \text{ and } AB_3 = \omega B_2.$$

Prove that $\|B_2\| = \|B_3\|$ and $\langle B_1, B_2\rangle = \langle B_1, B_3\rangle = \langle B_3, B_3\rangle = 0$. Hence (after rescaling B_2 and B_3 if necessary), $\{B_1, B_2, B_3\}$ is an orthogonal basis for $\mathbb{R}^3$ with respect to the standard scalar product.

Hint: *Show that $X^tAX = 0$ for any vector; then show that for $i \neq j$, the scalar product $\langle B_i, B_j\rangle$ can be expressed as a multiple of X^tAX for $X = B_i$ or $X = B_j$.*

(iv) Use these facts to prove that

$$R_t = \exp(tA) = B \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \omega t \\ 0 & -\omega t & 0 \end{pmatrix} B^t = B \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega t) & \sin(\omega t) \\ 0 & -\sin(\omega t) & \cos(\omega t) \end{pmatrix} B^t\,.$$

In particular, R_t is a clockwise rotation by ωt radians about the vector B_1:

$$R_t B_1 = B_1, \quad R_t(B_2) = \cos(\omega t)B_2 - \sin(\omega t)B_3\; R_t(B_3) = \sin(\omega t)B_2 + \cos(\omega t)B_3\,.$$

An old Math 136 midterm

(A1) Consider the non-zero 3×3 skew-symmetric matrix $A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$. Show that the spectrum of A is

$$\{0, i\sqrt{a^2 + b^2 + c^2}, -i\sqrt{a^2 + b^2 + c^2}\}\,.$$

(A2) Let $\langle\ ,\ \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the scalar product on $\mathbb{R}^2$ defined by

$$\langle X, Y\rangle = X^t \begin{pmatrix} a & b \\ b & c \end{pmatrix} Y\,.$$

Show that if $a < 0$ then the scalar product is not positive definite.

(A3) Find an orthonormal basis for the space of solutions of the equation $x - y + z = 0$.

(A4) Consider the matrix $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$. Its eigenvalues are $\lambda_1 = 7$ and $\lambda_2 = 2$ with corresponding eigenvectors $B_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $B_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Use this information to solve the initial value problem

$$\frac{dx_1}{dt} = 3x_1 + 2x_2 \qquad \frac{dx_2}{dt} = 2x_1 + 6x_2\,, \quad x_1(0) = 2\,, \quad x_2(0) = -3\,.$$

(A5) Let $\langle\ ,\ \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the scalar product on $\mathbb{R}^n$ defined by $\langle X, Y\rangle = X^t AY$, where A is a symmetric $n \times n$ matrix. Show that the scalar product is positive definite if and only if all of its eigenvalues are positive. **Hint:** *Use the fact that there is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.*

Another old Math 136 midterm

(B1) Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}$. Find an orthogonal basis for the kernel of the linear map

$$L_A : \mathbb{R}^5 \to \mathbb{R}^2 \ : \ X \mapsto AX.$$

(B2) Let A be a 3×3 symmetric matrix. You are told that $\lambda_1 = 3$, $\lambda_2 = -3$, and $\lambda_3 = 0$ are eigenvectors of A and that

$$B_1 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad B_2 = \frac{1}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \quad B_3 = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix},$$

are the corresponding (unit length) eigenvectors. Find A.

(B3) Suppose that $\langle\ ,\ \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is the scalar product on $\mathbb{R}^2$ given by

$$\langle X, Y\rangle = X^t \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix} Y\,.$$

for all $X, Y \in \mathbb{R}^2$. For what values of b is the scalar product $\langle\ ,\ \rangle$ positive definite?

(B4) Solve the initial value problem

$$\begin{aligned} x_1' &= x_2 + x_3, \qquad & x_1(0) &= 3 \\ x_2' &= x_1 + x_3, \qquad & x_2(0) &= 0 \\ x_3' &= x_1 + x_2, \qquad & x_3(0) &= 0 \end{aligned}$$

by finding an orthonormal basis of eigenvectors of the appropriate matrix.

Hint: $t^3 - 3t - 2 = (t-2)(t+1)^2$

(B5) Let A be an $n \times n$, skew-symmetric matrix. Use determinants to show that A is not invertible when n is odd.