Midterm review problems

Do the following problems in Lang:

p.61#21, p.61#22, p.70#6f, p.77#3, p.85#1,
p.93#4, p.93#5, p.109#3, p.110#15, p.121#1,
p.122#3, p.134#5, p.135#10, p.135#12, p.135#13,
p.136#18, p.136#19, p.141#8, p143#15, p144#16,
p.149#6, p.157#3, p.157#4, p.163#4, p.163#5,
p.163#7, p.169#11, p.170#13.

1. Let $A = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 4 & 4 & 4 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$

   (a) Find a basis for the kernel of the linear map $L_A : \mathbb{R}^5 \to \mathbb{R}^4$.

   (b) Find a basis for the image of the linear map $L_A : \mathbb{R}^5 \to \mathbb{R}^4$.

2. Find the inverse of the matrix $A = \begin{pmatrix} a & a & 0 \\ 0 & b & b \\ 0 & 0 & c \end{pmatrix}$, where a, b, c are non-zero numbers.

3. Let W be a subset of the vector space V.

   (a) What conditions must W satisfy for it to be a linear subspace of V?

   (b) Let V be the vector space of continuous functions on the closed inverval $[0, 1]$ and

   $$W = \{f \in V \;:\; f(0) = f(1) = 0\}\,.$$

   Prove that W is a linear subspace of V.

4. Let V be the three-dimensional vector space of polynomials of degree at most 2. The set $\{P_1, P_2, P_3\}$, where

   $$P_1(x) = 1\,, \quad P_2(x) = x\,, \quad P_3(x) = x^2\,,$$

   is a basis for V. (*You do not have to show this.*).

   Let $L : V \to V$ be the linear map given by the formula

   $$L(f) = f'' + f' + f\,.$$

   Find the matrix associated to L with respect to the above basis.

5. Let $L : V \to V$ be a linear map and suppose that $L(u) = u$ and $L(v) = 2v$, where u and v are non-zero vectors in V. Prove that $\{u, v\}$ is an independent set.

An old Math 136 midterm

1. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

   (a) Find a basis for the column space of A.

   (b) Find a basis for the row space of A.

2. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 2 \\ 5 & 5 & 1 & 2 & 8 \\ 3 & 3 & 1 & 1 & 5 \end{pmatrix}$ and let $L_A : \begin{cases} \mathbb{R}^5 & \to \mathbb{R}^4 \\ X & \mapsto A \cdot X \,. \end{cases}$

   (a) What is the dimension of $\mathrm{Im} L_A$ (the image of L_A)?

   (b) What is the dimension of $\mathrm{Ker} L_A$ (the kernel of L_A)?

   (c) Find a basis for $\mathrm{Ker} L_A$.

   (d) Find a basis for $\mathrm{Im} L_A$.

   (d) Find a basis for the row space of A.

3. Let $A_1, A_2 \in \mathbb{R}^5$ be non-zero column vectors satisfying the condition $A_1^t \cdot A_2 = 0$. Show that the 5×5 matrix

$$A = A_1 \cdot A_1^t + A_2 \cdot A_2^t$$

   has rank 2. **Hint:** *There is an obvious basis for the column space of A.*

4. Suppose that $\{v_1, v_2, v_3, v_4\}$ is a basis for the vector space V. Suppose further that $L : V \to V$ is a linear map whose kernel is generated by $\{v_1, v_2, v_3\}$. Finally, suppose that $L(v_4) = \lambda v_4$, for $\lambda \in \mathbb{R}$.

   Find the matrix of L with respect to the basis $\{v_1, v_2, v_3, v_4\}$.

5. Let $V = C^\infty(\mathbb{R})$ (i.e. the vector space of infinitely differentiable functions), and let $L : V \to V$ be the map defined by

$$L : f(x) \mapsto \frac{1}{2}(f(x) + f(-x))$$

   (a) Show that L is a linear map.

   (b) Describe the image of L.

   (c) Describe the kernel of L.