1. Consider the function

$$f(x,y) = 3x^2 + 2xy + 3y^2 = \begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Consider the new variables (X, Y) where

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

For an appropriate choice of θ, the function f will assume the form $f(x,y) = aX^2 + bY^2$. Find a, b, and θ by diagonalizing the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

Matrix exponentiation. Suppose that A is an $n \times n$ matrix. Let I be the $n \times n$ identity matrix, and define e^A to be

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k.$$

(For an $n \times n$ matrix A, define A^0 to be I. Also, it is possible to show that this infinite series always converges.)

2. (a) Prove that if A is diagonal with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, then e^A is diagonal with diagonal entries $e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}$.
   (b) Prove that if B is invertible, then for any positive integer k, $B^{-1}A^kB = (B^{-1}AB)^k$.
   (c) Use (b) to prove that if B is invertible, then $e^{B^{-1}AB} = B^{-1}e^AB$.

By the previous problem, $e^A = Be^{B^{-1}AB}B^{-1}$, and if $B^{-1}AB$ is diagonal, it is easy to compute $e^{B^{-1}AB}$. This gives a good way of computing matrix exponentials for diagonalizable matrices.

3. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ and show that $e^A = \begin{pmatrix} e & e^2 - e & e^3 - e^2 \\ 0 & e^2 & e^3 - e^2 \\ 0 & 0 & e^3 \end{pmatrix}$.

4. Let A be a symmetric $n \times n$ matrix. Prove that

$$\det\left(e^A\right) = e^{\mathrm{tr}(A)}$$