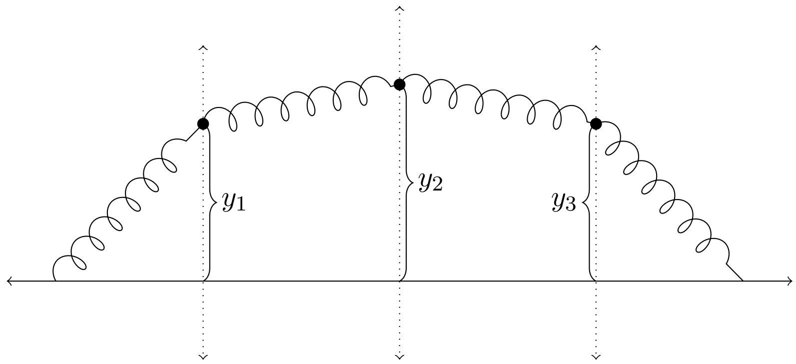
(1) Three objects of mass m connected by springs are free to move up and down as shown in the figure below.



Let y_k, for $k = 1, 2, 3$, denote the height of the kth object above the x-axis. Let F_k denote the vertical component of the net force on the kth object. One can show that if all the heights y_k are small, then

$$F_1 = -2Ky_1 + Ky_2\,, \quad F_2 = Ky_1 - 2Ky_2 + Ky_3\,, \quad F_3 = Ky_2 - 2Ky_3\,.$$

Let $\omega = \sqrt{K/m}$.

(a) Show that Newton's equations of motion can be written in the matrix form

$$\frac{d^2Y}{dt^2} + AY = O,$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \text{ and } A = \omega^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

(b) Find the general solution by diagonalizing the appropriate matrix.

(c) Interpret the eigenvectors you find.

(d) Suppose that at time $t = 0$, $y_1 = 1$, $y_2 = 0$, $y_3 = -1$, and the velocity of every mass is zero. Find $Y(t)$ for all t.

(2) Let A be a real $n \times n$ matrix and consider the linear map $L_A : \mathbb{R}^n \to \mathbb{R}^n$ given by multiplication by A.

Suppose that $\lambda = \alpha + i\beta \in \mathbb{C}$, $\beta \neq 0$, is a complex eigenvalue of L_A, with complex eigenvector $Z = X_1 + iX_2$, where X_1 and X_2 are column vectors in $\mathbb{R}^n$ (not both zero). Thus,

$$AZ = (AX_1) + i(AX_2) = (\alpha X_1 - \beta X_2) + i(\alpha X_2 + \beta X_1)\,.$$

(a) Prove that the vectors X_1 and X_2 are linearly independent and, therefore, span a 2-dimensional subspace $W = \text{span}(X_1, X_2) \subset \mathbb{R}^n$.

(b) Show that L_A restricts to define a linear map $L_W : W \to W$ and that the matrix of L_W with respect to the basis (X_1, X_2) of W is

$$|\lambda| \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

where $\lambda = |\lambda| e^{i\theta}$ (the polar form of λ).

(c) What is the geometrical interpretation of the result of part (b)?